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Contraction Options and Optimal Multiple-Stopping in Spectrally Negative Levy Models

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ABSTRACT. This paper studies the Lévy model of the optimal multiple-stopping problem arising in the context of the timing option to withdraw from a project in stages. The profits are driven by a general exponential spectrally negative Lévy process. This allows the model to incorporate sudden declines of the project values, generalizing greatly the classical geometric Brownian motion model. We solve the one-stage case as well as the extension to the multiple-stage case. The optimal stopping times are of threshold-type and the value function admits an expression in terms of the scale function. A series of numerical experiments are conducted to verify the optimality and to evaluate the efficiency of the algorithm.

1. INTRODUCTION

Consider a firm facing a decision of when to abandon or contract a project so as to maximize the total expected future cash flows. This problem is often referred to as the *abandonment option* or the *contraction option*. A typical formulation reduces to a standard optimal stopping problem, where the uncertainty of the future cash flow is driven by a stochastic process and the objective is to find a stopping time that maximizes the total expected cash flows realized until then. A more realistic extension is its multiple-stage version where the firm can withdraw from a project in stages.

In a standard formulation, given a discount rate $r > 0$ and $X_t = x + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$ for a standard Brownian motion W, $\mu \in \mathbb{R}$ and $\sigma > 0$, one wants to obtain a stopping time τ of X that maximizes the expectation

$$(1.1) \qquad \mathbb{E}\left[\int_0^\tau e^{-rt}(e^{X_t} - \delta)\mathrm{d}t + e^{-r\tau}K 1_{\{\tau < \infty\}} \right].$$

The profit collected continuously is modeled as the geometric Brownian motion e^{X_t} less the constant operating expense $\delta \geq 0$. The value $K \in \mathbb{R}$ corresponds to the lump-sum benefits attained (or the costs incurred) at the time of abandonment. Here a technical assumption $r > \mu$ is commonly imposed so that the expectation is finite and the problem is non-trivial. The problem is rather simple mathematically; it reduces to the well-known perpetual American option (or the McKean optimal stopping problem). An explicit solution can be attained even when X is generalized to a Lévy process (see, e.g., Mordecki [26]).

In this paper, we generalize the classical model by extending from Brownian motion to a general Lévy process with negative jumps (spectrally negative Lévy process), and consider the optimal stopping problem of the form:

$$
(1.2) \qquad \sup_{\tau} \mathbb{E} \left[\int_0^{\tau} e^{-rt} f(X_t) \mathrm{d}t + e^{-r\tau} g(X_{\tau}) 1_{\{\tau < \infty\}} \right].
$$

We obtain the optimal stopping time as well as the value function for the case f is increasing and g admits the form $g(x) = K - \sum_{i=1}^{N} c_i e^{a_i x}$ for some strictly positive constants c and a. We also show the optimality among all *stopping times of threshold type* (see (2.4) below) when g is relaxed to be a general decreasing and concave function.

We further extend to the multiple-stage case where one wants to obtain a set of stopping times $\{\tau^{(m)}; 1 \leq m \leq M\}$ such that $0 = \tau^{(0)} \leq \tau^{(1)} \leq \cdots \leq \tau^{(M)}$ a.s. and achieve

$$
(1.3) \qquad \sup_{\tau^{(1)} \leq \cdots \leq \tau^{(M)}} \sum_{m=1}^{M} \mathbb{E} \left[\int_{\tau^{(m-1)}}^{\tau^{(m)}} e^{-rt} F_m(X_t) \mathrm{d}t + e^{-r\tau^{(m)}} g_m(X_{\tau^{(m)}}) 1_{\{\tau^{(m)} < \infty\}} \right]
$$

when g_m and $f_m := F_m - F_{m+1}$ (with $F_{M+1} = 0$), for each $1 \leq m \leq M$, satisfy the same assumptions as in the one-stage case. The multiple-stopping problem arises frequently in real options (see e.g. [14]) and is well-studied particularly for the case X is driven by Brownian motion. In mathematical finance, similar problems are dealt in the valuation of swing options [10, 11].

Although the use of the geometric Brownian motion is fairly common in real options, empirical evidence suggests that the real world is not Gaussian, but with significant skewness and kurtosis (see, e.g., [8, 13, 30]). Dixit and Pindyck [14] considered the case with jumps of a fixed size with Poisson arrivals. Boyarchenko and Levendorskiĭ [9] considered the EPV approach for a general Lévy process satisfying the (ACP)-condition (with a focus on exponential-type jumps for illustration); they solved a related multiple-stage problem with g being constant. The Lévy model is in general less intractable than the continuous diffusion counterpart, especially when the lump-sum reward function g is not a constant. When jumps are involved, the process can potentially jump over a threshold level, requiring one to compute the overshoot/undershoot distributions that depend significantly on the form of the Lévy measure. Technical details are further required when it has jumps of infinite activity or infinite variation.

In this paper, we take advantage of the recent advances in the theory of the spectrally negative Lévy process (see, e.g., [7, 21]). In particular, we use the results by Egami and Yamazaki [17], where we obtained and showed the equivalence of the continuous/smooth fit condition and the first-order condition in a general optimal stopping problem. Unlike the two-sided jump case, the identification of the candidate optimal stopping time can be conducted efficiently without intricate computation. The resulting value function can be written in terms of the scale function, which also can be computed efficiently by using, e.g., [18, 29]. The extension to the multiple-stage can be carried out without losing generality. The resulting optimal stopping times are of threshold type with possibly simultaneous jumps, and the value function again admits the form in terms of the scale function. We also conduct a series of numerical experiments using the spectrally negative Lévy process with hyperexponential jumps so as to verify the optimality of the proposed strategies and also the efficiency of the proposed algorithm. We refer the reader to, among others, [1, 4, 5, 6, 15, 16, 22, 25] for optimal stopping problems of spectrally negative Lévy processes.

The rest of the paper is organized as follows. In Section 2, we review the spectrally negative Lévy process and the scale function and then solve the one-stage problem. In Section 3, we extend it to the multiple-stage problem.

In Section 4, we verify the optimality and efficiency of the algorithm through a series of numerical experiments. Section 5 concludes the paper.

2. ONE-STAGE PROBLEM

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a *spectrally negative Lévy process* $X = \{X_t : t \geq 0\}$ characterized uniquely by the *Laplace exponent*

$$(2.1) \qquad \psi(\beta) := \mathbb{E}^0\left[e^{\beta X_1}\right] = c\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(0,\infty)} (e^{-\beta z} - 1 + \beta z 1_{\{0<z<1\}}) \Pi(\mathrm{d}z), \quad \beta \in \mathbb{R},$$

where $c \in \mathbb{R}$, $\sigma \geq 0$ and Π is a Lévy measure concentrated on $(0, \infty)$ such that

$$(2.2) \qquad \int_{(0,\infty)} (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

Here and throughout the paper $\mathbb{P}^x$ is the conditional probability where $X_0 = x \in \mathbb{R}$ and $\mathbb{E}^x$ is its expectation. We exclude the case X is a negative subordinator (decreasing a.s.) and we shall further assume that the Lévy measure is *atomless*:

Assumption 2.1. *We assume that Π does not have atoms.*

This section considers the *one-stage optimal stopping* problem of the form (1.2) where the supremum is taken over the set (or a subset) of stopping times with respect to the natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by X. We assume the running payoff function f to be *locally bounded* and *increasing*. The stochastic process X models the state of the project and the monotonicity of f means that it yields higher benefits when X is high. Typically one assumes $f(x) = e^x - \delta$ as in (1.1) and this is clearly a special case of this model. Regarding the terminal reward function g, we consider two cases: (i) when g is a sum of exponential functions (Assumption 2.2 below) and (ii) when g is a general decreasing and concave function (Assumption 2.3 below).

The results discussed in this section are applications of Egami and Yamazaki [17] and will be extended to the multiple-stage problem in the next section. Let $\mathcal{S}$ be the set of all $[0, \infty]$-valued $\mathbb{F}$-stopping times and define for any $\tau \in \mathcal{S}$,

$$(2.3) \qquad u(x, \tau) \equiv u(x, \tau; f, g) := \mathbb{E}^x\left[\int_0^\tau e^{-rt} f(X_t)\mathrm{d}t + e^{-r\tau} g(X_\tau) 1_{\{\tau < \infty\}}\right], \quad x \in \mathbb{R}.$$

After a brief review on the scale function and the results of [17], we shall solve under Assumption 2.2 below that

$$u(x) := \sup_{\tau \in \mathcal{S}} u(x, \tau).$$

We then obtain under Assumption 2.3 below a weaker version of optimality

$$\widetilde{u}(x) := \sup_{\tau \in \widetilde{\mathcal{S}}} u(x, \tau),$$

over the set of all *first down-crossing times*,

$$\widetilde{\mathcal{S}} := \{\tau_A : A \in \mathbb{R}\},$$

where

$$(2.4) \qquad \tau_A := \inf\{t > 0 : X_t \leq A\}, \quad A \in \mathbb{R},$$

with $\inf \emptyset = \infty$ by convention. This form of optimality is often used in real options and also in the field of corporate finance and credit risk as exemplified by Leland's endogenous default model [23, 24]. In practice, a strategy must be simple enough to implement and it is in many cases a reasonable assumption to focus on the set of stopping times of threshold type as in (2.4). Because $\widetilde{\mathcal{S}} \subset \mathcal{S}$, it is clear that $u \geq \widetilde{u}$. For the rest of the paper, let $h_\pm(x) := \pm h(x) \vee 0$, $x \in \mathbb{R}$, for any measurable function $h : \mathbb{R} \mapsto \mathbb{R}$.

2.1. **Review of scale functions and Egami and Yamazaki** [17]. For any spectrally negative Lévy process, there exists a function called the *(r-)scale function*

$$W^{(r)} : \mathbb{R} \mapsto \mathbb{R}, \quad r \geq 0,$$

which is zero on $(-\infty, 0)$, continuous and strictly increasing on $[0, \infty)$, and is characterized by the Laplace transform:

$$\int_0^\infty e^{-sx} W^{(r)}(x) \mathrm{d}x = \frac{1}{\psi(s) - r}, \quad s > \Phi_r,$$

where

$$\Phi_r := \sup\{\lambda \geq 0 : \psi(\lambda) = r\}, \quad r \geq 0.$$

Here, the Laplace exponent ψ in (2.1) is known to be zero at the origin, convex on $\mathbb{R}_+$; Φ_r is well-defined and is strictly positive whenever $r > 0$. We also define

$$Z^{(r)}(x) := 1 + r \int_0^x W^{(r)}(y) \mathrm{d}y, \quad x \in \mathbb{R},$$

which is also called the scale function. As we shall see below, the pair of scale functions $W^{(r)}$ and $Z^{(r)}$ play significant roles in our problems.

As in Lemmas 8.3 and 8.5 of Kyprianou [21], for each $x > 0$, the functions $r \mapsto W^{(r)}(x)$ and $r \mapsto Z^{(r)}(x)$ can be analytically extended to $r \in \mathbb{C}$. Fix $a \geq 0$ and define $\psi_a(\cdot)$, as the Laplace exponent of X under $\mathbb{P}_a$ with the change of measure $\frac{\mathrm{d}\mathbb{P}_a}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t} = \exp(aX_t - \psi(a)t)$, $t \geq 0$; as in page 213 of [21], for all $\beta > -a$,

$$(2.5) \quad \psi_a(\beta) := \left(a\sigma^2 + c - \int_0^1 u(e^{-au} - 1)\Pi(\mathrm{d}u) \right)\beta + \frac{1}{2}\sigma^2\beta^2 + \int_0^\infty (e^{-\beta u} - 1 + \beta u 1_{\{0 < u < 1\}})e^{-au} \Pi(\mathrm{d}u).$$

If W_a and Z_a are the scale functions associated with X under $\mathbb{P}_a$ (or equivalently with $\psi_a(\cdot)$). Then, by Lemma 8.4 of [21],

$$(2.6) \quad\quad\quad\quad W_a^{(r-\psi(a))}(x) = e^{-ax}W^{(r)}(x), \quad x \geq 0.$$

In particular, by setting $a = \Phi_r$ (or equivalently $r = \psi(a)$), we can define

$$(2.7) \quad\quad\quad\quad W_{\Phi_r}(x) := W_{\Phi_r}^{(0)}(x) = e^{-\Phi_r x}W^{(r)}(x), \quad x \in \mathbb{R}$$

which satisfies

$$\int_0^\infty e^{-\beta x} W_{\Phi_r}(x) \mathrm{d}x = \frac{1}{\psi(\beta + \Phi_r) - r}, \quad \beta > 0.$$

The smoothness and asymptotic behaviors around zero of the scale function are particular important in our analysis. We summarize these in the remark given immediately below.

Remark 2.1. (1) *Assumption 2.1 guarantees that $W^{(r)}$ is C^1 on $(0,\infty)$. In particular, when $\sigma > 0$, then $W^{(r)}$ is C^2 on $(0,\infty)$. Fore more details on the smoothness of the scale function, see Chan et al. [12].*

(2) *As in Lemmas 4.3 and 4.4 of [22],*

$$W^{(r)}(0) = \left\{ \begin{array}{ll} 0, & \text{unbounded variation} \\ \frac{1}{\mu}, & \text{bounded variation} \end{array} \right\},$$

$$W^{(r)'}(0+) := \lim_{x \downarrow 0} W^{(r)'}(x) = \left\{ \begin{array}{ll} \frac{2}{\sigma^2}, & \sigma > 0 \\ \infty, & \sigma = 0 \text{ and } \Pi(0,\infty) = \infty \\ \frac{r+\Pi(0,\infty)}{\mu^2}, & \text{compound Poisson} \end{array} \right\},$$

where $\mu := c + \int_{(0,1)} z\,\Pi(\mathrm{d}z)$, which is finite when X is of bounded variation.

In [17], we have shown that a candidate optimal stopping time can be efficiently identified using the scale function. Define the expected payoff corresponding to the down-crossing time (2.4) by

$$(2.8) \qquad u_A(x) := u(x, \tau_A), \quad x, A \in \mathbb{R},$$

which necessarily equals $g(x)$ for $x \leq A$. By combining the compensation formula for Lévy processes and the resolvent measure written in terms of the scale function, this can be written in a semi-explicit form. Let

$$(2.9) \qquad \Psi_f(A) := \int_0^\infty e^{-\Phi_r y} f(y+A)\mathrm{d}y, \quad A \in \mathbb{R},$$

$$(2.10) \qquad \Theta_f(x;A) := \left\{ \begin{array}{ll} \int_A^x W^{(r)}(x-y)f(y)\mathrm{d}y, & x > A, \\ 0, & x \leq A, \end{array} \right.$$

and

$$\rho_{g,A}^{(r)} := \int_0^\infty \Pi(\mathrm{d}u) \int_0^u e^{-\Phi_r z}(g(z+A-u) - g(A))\mathrm{d}z$$

$$(2.11) \qquad \equiv \int_0^\infty \Pi(\mathrm{d}u) \int_A^{u+A} e^{-\Phi_r(y-A)}(g(y-u) - g(A))\mathrm{d}y, \quad A \in \mathbb{R},$$

$$\varphi_{g,A}^{(r)}(x) := \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u \wedge (x-A)} W^{(r)}(x-z-A)(g(z+A-u) - g(A))\mathrm{d}z, \quad x > A.$$

These integrals are well-defined if

$$(2.12) \qquad \int_0^\infty e^{-\Phi_r y}|f(y)|\mathrm{d}y < \infty,$$

$$(2.13) \qquad g \in C^2, \quad \text{and} \quad \int_1^\infty \Pi(\mathrm{d}u) \max_{A-u \leq \zeta \leq A} |g(\zeta) - g(A)| < \infty, \quad A \in \mathbb{R}.$$

If these are satisfied, we can write $u_A(x)$ as in (2.8) for $x > A$ as the sum of the following three terms:

$$\Gamma_1(x;A) := g(A)\left[Z^{(r)}(x-A) - \frac{r}{\Phi_r}W^{(r)}(x-A)\right],$$

$$(2.14) \qquad \Gamma_2(x;A) := W^{(r)}(x-A)\rho_{g,A}^{(r)} - \varphi_{g,A}^{(r)}(x),$$

$$\Gamma_3(x;A) := W^{(r)}(x-A)\Psi_f(A) - \Theta_f(x;A).$$

Egami and Yamazaki [17] obtained the *first-order condition* that makes $\partial u_A(x)/\partial A$ vanish and showed that it is equivalent to the *continuous fit* condition $u_A(A+) := \lim_{x \downarrow A} u_A(x) = g(A)$ when X is of bounded variation and to the *smooth fit* condition $u'_A(A+) := \lim_{x \downarrow A} u'_A(x) = g'(A)$ when $\sigma > 0$. Recall that X is of bounded variation if and only if $\sigma = 0$ and

$$(2.15) \qquad \int_{(0,\infty)} (1 \wedge z)\, \Pi(\mathrm{d}z) < \infty.$$

It has been shown that

$$(2.16) \qquad u_A(A+) = g(A) + W^{(r)}(0)\Lambda(A), \quad A \in \mathbb{R},$$

where

$$(2.17) \qquad \Lambda(A) \equiv \Lambda(A; f, g) := -\frac{r}{\Phi_r} g(A) - \frac{\sigma^2}{2} g'(A) + \rho_{g,A}^{(r)} + \Psi_f(A), \quad A \in \mathbb{R}.$$

In view of Remark 2.1 (2), for the unbounded variation case, continuous fit holds whatever the choice of A is, while, for the bounded variation case, it holds if and only if $\Lambda(A) = 0$.

Furthermore, it has been shown, on condition that there exists some $\delta > 0$ satisfying

$$(2.18) \qquad \int_1^\infty \Pi(\mathrm{d}u) \sup_{0 \leq \xi \leq \delta} |g(A + \xi) - g(A + \xi - u)| < \infty,$$

we have

$$(2.19) \qquad \frac{\partial}{\partial A} u_A(x) = -e^{\Phi_r(x-A)} W'_{\Phi_r}(x - A)\Lambda(A), \quad x > A,$$

where W_{Φ_r} is defined in (2.7). It is known that W_{Φ_r} is increasing and hence, if $\Lambda(A)$ is monotonically increasing, the down-crossing time τ_A for the A with $\Lambda(A) = 0$ becomes a natural candidate for the optimal stopping time.

2.2. **Exponential Case.** We first consider the case where g admits the form:

$$(2.20) \qquad g(x) = K - \sum_{i=1}^N c_i e^{a_i x}, \quad x \in \mathbb{R},$$

for some constants $K \in \mathbb{R}$ and $c_i, a_i > 0$, $1 \leq i \leq N$. We assume without loss of generality that $a_i \neq a_j$ for $i \neq j$. Because g is bounded on $(-\infty, B)$ for any $B \in \mathbb{R}$, (2.13) and (2.18) are clearly satisfied. For f, we need a technical condition so that (2.12) is guaranteed.

Assumption 2.2. *Suppose*

(1) *$f(\cdot)$ is continuous, increasing, $f(-\infty) := \lim_{x \downarrow -\infty} f(x) > -\infty$ and $\int_0^\infty e^{-\Phi_r y} f_+(y)\mathrm{d}y < \infty$;*
(2) *$g(\cdot)$ admits the form (2.20) for some $K \in \mathbb{R}$ and strictly positive constants a_i and c_i, $1 \leq i \leq N$, such that $a_i \neq a_j$ for any $i \neq j$.*

Remark 2.2. *For the special case $r > \max_{1 \leq i \leq N} \psi(a_i)$ or equivalently $\Phi_r > \max_{i=1,...N} a_i$, we can simplify the problem and rewrite it so that g is a constant and f is an increasing function.*

With Assumption 2.2, we simplify (2.17) using

$$(2.21) \qquad \varpi_r(a) := \left\{ \begin{array}{ll} \frac{r-\psi(a)}{\Phi_r-a}, & a \neq \Phi_r \\ \psi'(\Phi_r) = \lim_{a \to \Phi_r} \frac{r-\psi(a)}{\Phi_r-a}, & a = \Phi_r \end{array} \right\}, \quad a > 0.$$

By the monotonicity of ψ, $\varpi_r(a) > 0$ for any $a > 0$. The proof of the following lemma is given in Appendix A.1.

Lemma 2.1. *For every $A \in \mathbb{R}$, we have*

$$(2.22) \qquad \Lambda(A) = -\frac{r}{\Phi_r} K + \sum_{i=1}^{N} c_i e^{a_i A} \varpi_r(a_i) + \Psi_f(A).$$

In view of (2.22) above, the function $\Lambda(A)$ is clearly continuous and increasing. Therefore, if $\lim_{A\downarrow-\infty} \Lambda(A) < 0 < \lim_{A\uparrow\infty} \Lambda(A)$, there exists a unique root $A^* \in \mathbb{R}$ such that $\Lambda(A^*) = 0$. Otherwise, let $A^* = -\infty$ if $\lim_{A\downarrow-\infty} \Lambda(A) \geq 0$ and let $A^* = \infty$ if $\lim_{A\uparrow\infty} \Lambda(A) \leq 0$.

Remark 2.3. (1) *For any stopping time $\tau \in \mathcal{S}$, recall Assumption 2.2 (1) and notice $\mathbb{E}^x \left[\int_0^\tau e^{-rt} f(-\infty) dt \right] = \frac{f(-\infty)}{r}(1 - \mathbb{E}^x[e^{-r\tau}])$. Therefore, with $\widetilde{f}(x) := f(x) - f(-\infty) \geq 0$ and $\widetilde{g}(x) := g(x) - \frac{f(-\infty)}{r}$,*

$$u(x) = \frac{f(-\infty)}{r} + \sup_{\tau \in \mathcal{S}} u(x, \tau; \widetilde{f}, \widetilde{g}).$$

Because $\widetilde{g}$ is decreasing, there exists a unique $\widetilde{A} \in [-\infty, \infty]$ such that $\widetilde{g}(x) < 0 \iff x \in (\widetilde{A}, \infty)$. Because $\widetilde{f}$ is nonnegative, it is easy to see that it is never optimal to stop on $(\widetilde{A}, \infty)$ and hence we can also write

$$u(x) = \frac{f(-\infty)}{r} + \sup_{\tau \in \mathcal{S}} u(x, \tau; \widetilde{f}, \widetilde{g}_+).$$

(2) *Applying (2.17) to the pair $\widetilde{f}$ and $\widetilde{g}$, because $\widetilde{f}$ is uniformly positive,*

$$\Lambda(A; \widetilde{f}, \widetilde{g}) = -\frac{r}{\Phi_r} \left(K - \frac{f(-\infty)}{r} \right) + \sum_{i=1}^{N} c_i e^{a_i A} \varpi_r(a_i) + \Psi_{\widetilde{f}}(A)$$

$$\geq -\frac{r}{\Phi_r} \left(K - \frac{f(-\infty)}{r} \right) + \sum_{i=1}^{N} c_i e^{a_i A} \varpi_r(a_i) \geq -\frac{r}{\Phi_r} \left[K - \frac{f(-\infty)}{r} - \sum_{i=1}^{N} c_i e^{a_i A} \right] = -\frac{r}{\Phi_r} \widetilde{g}(A),$$

where the second to last inequality holds because, by the convexity of ψ on $[0, \infty)$, $\varpi_r(a_i) > \frac{r}{\Phi_r}$. If $-\infty < \widetilde{A} < \infty$, because $\widetilde{A}$ is the unique root of $-\frac{r}{\Phi_r} \widetilde{g}(A) = 0$, we must have $A^* \leq \widetilde{A}$. If $\widetilde{A} = -\infty$, we have $\Lambda(A) \geq -\frac{r}{\Phi_r} \widetilde{g}(A) \geq 0$ for all $A \in \mathbb{R}$ and hence $A^* = -\infty$.

(3) *Except for the case g is a constant, because $\widetilde{g}$ decreases to $-\infty$, $\widetilde{A} < \infty$. Hence (2) implies $A^* < \infty$.*

When $-\infty < A^* < \infty$, the corresponding expected value becomes (see, e.g., Exercise 8.7 (ii) of [21])

$$u_{A^*}(x) = K \left(Z^{(r)}(x - A^*) - \frac{r}{\Phi_r} W^{(r)}(x - A^*) \right) - \sum_{i=1}^{N} c_i e^{a_i x} \left(Z_{a_i}^{(r-\psi(a_i))}(x - A^*) - \varpi_r(a_i) W_{a_i}^{(r-\psi(a_i))}(x - A^*) \right)$$

$$+ W^{(r)}(x - A^*) \Psi_f(A^*) - \Theta_f(x; A^*).$$

By how A^* is chosen in (2.22) and by (2.6), it can be simplified to

$$(2.23) \qquad u_{A^*}(x) = KZ^{(r)}(x - A^*) - \sum_{i=1}^{N} c_i e^{a_i x} Z_{a_i}^{(r - \psi(a_i))}(x - A^*) - \Theta_f(x; A^*).$$

The verification of optimality requires the following smoothness properties, whose proofs are given in Appendix A.2.

Lemma 2.2. *Suppose $-\infty < A^* \leq \infty$.*

(1) $u_{A^*}(x)$ *is C^1 on $\mathbb{R}\backslash\{A^*\}$.*

(2) *In particular, when X is of unbounded variation, $u_{A^*}(x)$ is C^2 on $\mathbb{R}\backslash\{A^*\}$.*

Herein, we add a remark concerning continuous/smooth fit. The following remark confirms the results in [17] and further verifies that smooth fit holds whenever X is of unbounded variation even when $\sigma = 0$. This observation only requires the asymptotic behavior of the scale function near zero as in Remark 2.1 (2).

Remark 2.4 (continuous/smooth fit).   (1) *Continuous fit holds (i.e. $u_{A^*}(A^*+) = g(A^*)$) because, by (2.23),* $Z^{(r)}(0) = Z_{a_i}^{(r - \psi(a_i))}(0) = 1$ *and $\lim_{x\downarrow A^*} \Theta_f(x; A^*) = 0$.*

(2) *In particular, when X is of unbounded variation, smooth fit holds (i.e. $u'_{A^*}(A^*+) = g'(A^*)$) because*

$$u'_{A^*}(x) = KrW^{(r)}(x - A^*) - \sum_{i=1}^{N} c_i e^{a_i x}(r - \psi(a_i))W_{a_i}^{(r - \psi(a_i))}(x - A^*)$$

$$- \sum_{i=1}^{N} a_i c_i e^{a_i x} Z_{a_i}^{(r - \psi(a_i))}(x - A^*) - \Theta'_f(x; A^*) \xrightarrow{x\downarrow A^*} - \sum_{i=1}^{N} a_i c_i e^{a_i A^*} = g'(A^*)$$

thanks to $W^{(r)}(0) = W_{a_i}^{(r - \psi(a_i))}(0) = 0$, $Z_{a_i}^{(r - \psi(a_i))}(0) = 1$ and $\lim_{x\downarrow A} \Theta'_f(x; A) = 0$; see also the proof of Lemma 2.2.

We now state the main results of this subsection. The proof is given in Appendix A.3.

Proposition 2.1.   (1) *If $-\infty < A^* < \infty$, the stopping time $\tau_{A^*} := \inf\{t \geq 0 : X_t \leq A^*\}$ is optimal over $\mathcal{S}$ and the value function is $u(x) = u_{A^*}(x)$ as in (2.23) for all $x \in \mathbb{R}$.*

(2) *If $A^* = \infty$, immediate stopping is always optimal and $u(x) = g(x)$ for any $x \in \mathbb{R}$.*

(3) *If $A^* = -\infty$, it is never optimal to stop (i.e. $\tau^* = \infty$ is optimal), and the value function is given by*

$$(2.24) \qquad u(x) = \int_{-\infty}^{\infty} \left(\Phi'_r e^{-\Phi_r(x-y)} - W^{(r)}(y - x)\right) f(y) \mathrm{d}y,$$

where Φ'_r is the derivative of Φ_r with respect to r.

2.3. **For a general concave and decreasing g.** We now relax the assumption on g and consider a general concave and decreasing function g. We also drop the continuity assumption on f.

Assumption 2.3. *Suppose*

(1) $f(\cdot)$ *is increasing such that $f(-\infty) := \lim_{x\downarrow-\infty} f(x) > -\infty$ and $\int_0^\infty e^{-\Phi_r y} f_+(y) \mathrm{d}y < \infty$;*

(2) $g(\cdot)$ *is twice-differentiable, concave and monotonically decreasing such that (2.13) and (2.18) hold.*

Under this assumption, we see that $\Lambda(A)$ is increasing. Indeed, we have

$$\frac{\partial}{\partial A}\left[-\frac{r}{\Phi_r}g(A) - \frac{\sigma^2}{2}g'(A) + \rho_{g,A}^{(r)}\right] = -\frac{r}{\Phi_r}g'(A) - \frac{\sigma^2}{2}g''(A) + \int_0^\infty \Pi(\mathrm{d}u)\int_0^u e^{-\Phi_r z}(g'(z+A-u) - g'(A))\mathrm{d}z.$$

This is positive because $g'(A) \leq 0$ and $g''(A) \leq 0$ for any $A \in \mathbb{R}$ and because by the concavity and monotonicity $z + A - u \leq A$ for $z < u$ and hence the integral is nonnegative. On the other hand, $\Psi_f(A)$ is increasing because f is. Therefore, we again define A^* in the same way as the unique root of $\Lambda(A) = 0$ (if it exists). The proof of the following result is given in Appendix A.4.

Proposition 2.2. *Suppose Assumption 2.3.*

    (1) *When $-\infty < A^* < \infty$, then τ_{A^*} is optimal over $\widetilde{\mathcal{S}}$ and the value function is given by*

$$(2.25) \quad \widetilde{u}(x) = u_{A^*}(x) = g(A^*)Z^{(r)}(x - A^*) + W^{(r)}(x - A^*)\frac{\sigma^2}{2}g'(A^*) - \varphi_{g,A^*}^{(r)}(x) - \Theta_f(x; A^*), \quad x > A^*.$$

    For $x \leq A^*$, we have $\widetilde{u}(x) = g(x)$.

    (2) *If $A^* = \infty$, immediate stopping is always optimal and $\widetilde{u}(x) = g(x)$ for any $x \in \mathbb{R}$.*

    (3) *If $A^* = -\infty$, then $\tau^* = \infty$ is optimal over $\mathcal{S}$ and (2.24) holds.*

3. MULTIPLE-STAGE PROBLEM

In this section, we extend to the scenario the firm can decrease its involvement in the project in multiple stages as defined in (1.3). As in the one-stage case, we consider two modes of optimality:

$$(3.1) \qquad U^{(M)}(x) := \sup_{(\tau^{(1)},\ldots,\tau^{(M)})\in\mathcal{S}_M} \sum_{m=1}^M \mathbb{E}^x\left[\int_{\tau^{(m-1)}}^{\tau^{(m)}} e^{-rt}F_m(X_t)\mathrm{d}t + e^{-r\tau^{(m)}}g_m(X_{\tau^{(m)}})1_{\{\tau^{(m)}<\infty\}}\right],$$

$$(3.2) \qquad \widetilde{U}^{(M)}(x) := \sup_{(\tau^{(1)},\ldots,\tau^{(M)})\in\widetilde{\mathcal{S}}_M} \sum_{m=1}^M \mathbb{E}^x\left[\int_{\tau^{(m-1)}}^{\tau^{(m)}} e^{-rt}F_m(X_t)\mathrm{d}t + e^{-r\tau^{(m)}}g_m(X_{\tau^{(m)}})1_{\{\tau^{(m)}<\infty\}}\right],$$

for all $x \in \mathbb{R}$ where we define $\tau^{(0)} := 0$ for notational brevity and the supremum is, respectively, over the set of increasing sequences of M stopping times,

$$\mathcal{S}_M := \{\tau^{(m)} \in \mathcal{S}, 1 \leq m \leq M : \tau^{(1)} \leq \cdots \leq \tau^{(M)}\},$$

and over the set of increasing sequences of M down-crossing times,

$$\widetilde{\mathcal{S}}_M := \{\tau^{(m)} = \tau_{A_m} \in \widetilde{\mathcal{S}}, 1 \leq m \leq M : A_1 \geq A_2 \cdots \geq A_M\}.$$

Clearly, $\widetilde{\mathcal{S}}_M \subset \mathcal{S}_M$ and hence $\widetilde{U}^{(M)} \leq U^{(M)}$.

We first consider the case g_m admits the form

$$(3.3) \qquad\qquad g_m(x) := K_m - \sum_{i=1}^{N_m} c_{mi}e^{a_{mi}x} \quad 1 \leq m \leq M,$$

for some constants $K_m \in \mathbb{R}$ and $c_{mi}, a_{mi} > 0$, $1 \leq i \leq N_m$, and show the optimality in the sense of (3.1) as an extension of Proposition 2.1. We then consider a more general case where g_m is twice-differentiable, concave

and monotonically decreasing and show the optimality over $\widetilde{\mathcal{S}}_M$ as an extension of Proposition 2.2. Regarding the running reward function F, define the differences:

$$f_m := F_m - F_{m+1}, \quad 1 \le m \le M,$$

with $F_{M+1} \equiv 0$. As is also assumed in [9], we consider the case f_m is increasing. Using the notation as in (2.3), we can then write for all $x \in \mathbb{R}$

$$(3.4) \qquad U^{(M)}(x) = \sup_{(\tau^{(1)},\ldots,\tau^{(M)})\in\mathcal{S}_M} \sum_{m=1}^{M} u(x,\tau^{(m)}; f_m, g_m),$$

$$(3.5) \qquad \widetilde{U}^{(M)}(x) = \sup_{(\tau^{(1)},\ldots,\tau^{(M)})\in\widetilde{\mathcal{S}}_M} \sum_{m=1}^{M} u(x,\tau^{(m)}; f_m, g_m).$$

In summary, we assume Assumptions 3.1 and 3.2 below for (3.1) and (3.2), respectively.

Assumption 3.1. *For each $1 \le m \le M$, we assume that f_m and g_m satisfy Assumption 2.2.*

Assumption 3.2. *For each $1 \le m \le M$, we assume that f_m and g_m satisfy Assumption 2.3.*

As is clear from the problem structure, simultaneous stoppings (i.e. $\tau_k = \cdots = \tau_{k+l}$ a.s. for some k and l) may be optimal in case it is not advantageous to stay in some intermediate stages (i.e. stages $k+1,\ldots,k+l$). For this reason, define, for any subinterval $\mathcal{I} = \{\min\mathcal{I}, \min\mathcal{I}+1, \ldots, \max\mathcal{I}\} \subset \{1,\ldots,M\}$,

$$(3.6) \qquad g_{\mathcal{I}} := \sum_{i\in\mathcal{I}} g_i \quad \text{and} \quad f_{\mathcal{I}} := F_{\min\mathcal{I}} - F_{\max\mathcal{I}+1},$$

and consider an *auxiliary one-stage problem* (1.2) with $g = g_{\mathcal{I}}$ and $f = f_{\mathcal{I}}$. Notice that Assumption 3.1 (resp. Assumption 3.2) guarantees that $f_{\mathcal{I}}$ and $g_{\mathcal{I}}$ also satisfy Assumption 2.2 (Assumption 2.3) for any $\mathcal{I}$. Hence Propositions 2.1 and 2.2 apply.

Let

$$(3.7) \qquad \Lambda_m(A) := \Lambda(A; f_m, g_m), \quad A \in \mathbb{R}, 1 \le m \le M,$$

as the function (2.17) for (f_m, g_m). Because $\rho^{(r)}_{h_1+h_2,A} \equiv \rho^{(r)}_{h_1,A} + \rho^{(r)}_{h_2,A}$ and $\Psi_{h_1+h_2}(A) \equiv \Psi_{h_1}(A) + \Psi_{h_2}(A)$ for any measurable functions h_1 and h_2, we see that

$$(3.8) \qquad \Lambda_{\mathcal{I}}(A) := \Lambda(A; f_{\mathcal{I}}, g_{\mathcal{I}}) = \Lambda\Big(A; \sum_{m\in\mathcal{I}} f_m, \sum_{m\in\mathcal{I}} g_m\Big) = \sum_{m\in\mathcal{I}} \Lambda_m(A)$$

is increasing and corresponds to the function (2.17) for $(f_{\mathcal{I}}, g_{\mathcal{I}})$. In particular, under Assumption 2.2, this reduces to

$$\Lambda_{\mathcal{I}}(A) = \sum_{m\in\mathcal{I}} \Big(-\frac{r}{\Phi_r} K_m + \sum_{i=1}^{N_m} c_{mi} e^{a_{mi}A} \varpi_r(a_{mi}) + \Psi_{f_m}(A) \Big).$$

Now let $A^*_{\mathcal{I}}$ be the root of $\Lambda_{\mathcal{I}}(A) = 0$ if it exists. If $\lim_{A\uparrow\infty} \Lambda_{\mathcal{I}}(A) \le 0$, we set $A^*_{\mathcal{I}} = \infty$; if $\lim_{A\downarrow-\infty} \Lambda_{\mathcal{I}}(A) \ge 0$, we set $A^*_{\mathcal{I}} = -\infty$. For simplicity, let $A^*_m := A^*_{\{m\}}$ for any $1 \le m \le M$. Also define

$$u^{\mathcal{I}}_A(x) := u(x, \tau_A; f_{\mathcal{I}}, g_{\mathcal{I}}), \quad x, A \in \mathbb{R}.$$

With these notations, the following is immediate by Propositions 2.1 and 2.2.

Corollary 3.1. *Fix any $\mathcal{I}$ and $x \in \mathbb{R}$, and consider the problems:*

$$u_{\mathcal{I}}(x) := \sup_{\tau \in \mathcal{S}} u(x, \tau; f_{\mathcal{I}}, g_{\mathcal{I}}) \quad \text{and} \quad \widetilde{u}_{\mathcal{I}}(x) := \sup_{\tau \in \widetilde{\mathcal{S}}} u(x, \tau; f_{\mathcal{I}}, g_{\mathcal{I}}).$$

Suppose Assumption 3.1.

(1) *If $-\infty < A_{\mathcal{I}}^* < \infty$, then*

$$u_{\mathcal{I}}(x) = u_{A_{\mathcal{I}}^*}^{\mathcal{I}}(x) = \sum_{m \in \mathcal{I}} \left(K_m Z^{(r)}(x - A_{\mathcal{I}}^*) - \sum_{i=1}^{N_m} c_{mi} e^{a_{mi}x} Z_{a_{mi}}^{(r - \psi(a_{mi}))}(x - A_{\mathcal{I}}^*) \right) - \Theta_{f_{\mathcal{I}}}(x; A_{\mathcal{I}}^*),$$

and the stopping time $\tau_{A_{\mathcal{I}}^*} := \inf\{t > 0 : X_t \leq A_{\mathcal{I}}^*\}$ is optimal.

(2) *If $A_{\mathcal{I}}^* = \infty$, $u_{\mathcal{I}}(x) = g_{\mathcal{I}}(x)$ for any $x \in \mathbb{R}$ with optimal stopping time $\tau^* = 0$.*

(3) *If $A_{\mathcal{I}}^* = -\infty$, it is never optimal to stop, and the value function is given by*

(3.9)
$$u_{\mathcal{I}}(x) = \int_{-\infty}^{\infty} \left(\Phi_r' e^{-\Phi_r(x-y)} - W^{(r)}(y - x) \right) f_{\mathcal{I}}(y) \mathrm{d}y.$$

Suppose Assumption 3.2.

(1) *If $-\infty < A_{\mathcal{I}}^* < \infty$, then*

$$\widetilde{u}_{\mathcal{I}}(x) = u_{A_{\mathcal{I}}^*}^{\mathcal{I}}(x) = g_{\mathcal{I}}(A_{\mathcal{I}}^*) Z^{(r)}(x - A_{\mathcal{I}}^*) + W^{(r)}(x - A_{\mathcal{I}}^*) \frac{\sigma^2}{2} g_{\mathcal{I}}'(A_{\mathcal{I}}^*) - \varphi_{g_{\mathcal{I}}, A_{\mathcal{I}}^*}^{(r)}(x) - \Theta_{f_{\mathcal{I}}}(x; A_{\mathcal{I}}^*), \quad x > A_{\mathcal{I}}^*,$$

with the optimal stopping time $\tau_{A_{\mathcal{I}}^*}$. For $x \leq A_{\mathcal{I}}^*$, we have $\widetilde{u}(x) = g(x)$.

(2) *If $A_{\mathcal{I}}^* = \infty$, $\widetilde{u}_{\mathcal{I}}(x) = g_{\mathcal{I}}(x)$ for any $x \in \mathbb{R}$ with optimal stopping time $\tau^* = 0$.*

(3) *If $A_{\mathcal{I}}^* = -\infty$, then $\tau^* = \infty$ is optimal over $\mathcal{S}$ and (3.9) holds.*

3.1. Two-stage problem.

In order to gain intuition, we first consider the case with $M = 2$ and obtain $U^{(2)}(x)$ and $\widetilde{U}^{(2)}(x)$ under Assumptions 3.1 and 3.2, respectively. Following the procedures discussed above, $A_m^* \in [-\infty, \infty]$, or the root of $\Lambda_m(A) = 0$, is well-defined for $m = 1, 2$. As a special case of (3.6),

(3.10)
$$f_2 \equiv F_2, \quad f_1 \equiv F_1 - F_2 \equiv F_1 - f_2, \quad \text{and} \quad f_{\{1,2\}} \equiv F_1 \equiv f_1 + f_2.$$

We shall consider the cases (i) $A_1^* > A_2^*$ and (ii) $A_1^* \leq A_2^*$, separately. For (i), we shall show that $(\tau_{A_1^*}, \tau_{A_2^*})$ is optimal. For (ii), we shall show that simultaneous stoppings are optimal. We first consider the former.

Proposition 3.1. *If $\infty \geq A_1^* > A_2^* \geq -\infty$, then $(\tau_{A_1^*}, \tau_{A_2^*})$ is optimal; the value function is given by $U^{(2)}(x) = \sum_{m=1,2} u_{A_m^*}^{\{m\}}(x)$ and $\widetilde{U}^{(2)}(x) = \sum_{m=1,2} u_{A_m^*}^{\{m\}}(x)$ under Assumptions 3.1 and 3.2, respectively. In particular, under Assumption 3.1 and if $\infty > A_1^* > A_2^* > -\infty$,*

(3.11)
$$U^{(2)}(x) = \sum_{m=1,2} \left(K_m Z^{(r)}(x - A_m^*) - \sum_{i=1}^{N_m} c_{mi} e^{a_{mi}x} Z_{a_{mi}}^{(r - \psi(a_{mi}))}(x - A_m^*) \right)$$
$$- \int_{A_1^*}^{x} W^{(r)}(x - y) F_1(y) \mathrm{d}y - \int_{A_2^*}^{A_1^*} W^{(r)}(x - y) F_2(y) \mathrm{d}y.$$

Proof. Suppose Assumption 3.1 holds. By relaxing the constraint that $\tau^{(1)} \leq \tau^{(2)}$, we can obtain an upper bound:

$$U^{(2)}(x) \leq \overline{U}^{(2)}(x) := \sum_{m=1,2} \sup_{\tau^{(m)} \in \mathcal{S}} u(x, \tau^{(m)}; f_m, g_m) = \sum_{m=1,2} u_{A_m^*}^{\{m\}}(x),$$

where the last equality holds by Corollary 3.1. On the other hand, because $\tau_{A_1^*} \leq \tau_{A_2^*}$ a.s. (hence $(\tau_{A_1^*}, \tau_{A_2^*}) \in \mathcal{S}_2$) thanks to $A_1^* > A_2^*$, we have $U^{(2)}(x) \geq \overline{U}^{(2)}(x)$, as desired. The same result holds under Assumption 3.2 by relaxing the constraint that $A_1 \geq A_2$ and noticing that $(\tau_{A_1^*}, \tau_{A_2^*}) \in \widetilde{\mathcal{S}}_2$.

For the second claim, because $A_1^* > A_2^*$ and by (3.10),

$$\sum_{m=1,2} \Theta_{f_m}(x, A_m^*) = \sum_{m=1,2} \int_{A_m^*}^x W^{(r)}(x-y) f_m(y) \mathrm{d}y = \int_{A_1^*}^x W^{(r)}(x-y) F_1(y) \mathrm{d}y + \int_{A_2^*}^{A_1^*} W^{(r)}(x-y) F_2(y) \mathrm{d}y,$$

and hence (3.11) holds in view of Corollary 3.1.

$\square$

Now consider the case $-\infty \leq A_1^* \leq A_2^* \leq \infty$.

Lemma 3.1. *Suppose* $-\infty \leq A_1^* \leq A_2^* \leq \infty$. *Under Assumption 3.1 (resp. Assumption 3.2), the first optimal stopping cannot occur on* (A_2^*, ∞); *namely if* $\tau^{*(1)}$ *is the optimal first stopping time in the sense of* (3.1) *((3.2)), then* $X_{\tau^{*(1)}} \in (-\infty, A_2^*]$ *a.s. on* $\{\tau^{*(1)} < \infty\}$.

Proof. The result is immediate when $A_2^* = \infty$ and hence we assume $A_2^* < \infty$.

Suppose Assumption 3.1 holds, and in order to derive a contradiction, we suppose there exists some $\hat{x} > A_2^*$ at which it is optimal to stop. Under this assumption, the value function must satisfy

$$(3.12) \qquad U^{(2)}(\hat{x}) = g_1(\hat{x}) + \sup_{\tau \in \mathcal{S}} u(\hat{x}, \tau; f_2, g_2) = g_1(\hat{x}) + u_{A_2^*}^{\{2\}}(\hat{x}).$$

We shall show that this is in fact smaller than $u_{A_2^*}^{\{1\}}(\hat{x}) + u_{A_2^*}^{\{2\}}(\hat{x})$, which is the value obtained by $(\tau^{(1)}, \tau^{(2)}) = (\tau_{A_2^*}, \tau_{A_2^*}) \in \widetilde{\mathcal{S}}_2 \subset \mathcal{S}_2$. By (2.16) and (2.19),

$$(3.13) \qquad \Lambda_1(A) \leq (\geq)0 \implies \frac{\partial}{\partial A} u_A^{\{1\}}(\hat{x}) \geq (\leq)0 \quad \forall A < \hat{x},$$

$$(3.14) \qquad u_A^{\{1\}}(A+) = g_1(A) + W^{(r)}(0)\Lambda_1(A).$$

By (3.13)-(3.14) and because $A_1^* \leq A_2^* < \hat{x}$ and Λ_1 is increasing,

$$u_{A_2^*}^{\{1\}}(\hat{x}) > \lim_{A \uparrow \hat{x}} u_A^{\{1\}}(\hat{x}) = g_1(\hat{x}) + W^{(r)}(0)\Lambda_1(\hat{x}) \geq g_1(\hat{x}).$$

Regarding the last inequality, for the unbounded variation case, it holds because $W^{(r)}(0) = 0$ by Remark 2.1 (2). For the bounded variation case, it also holds because (3.13) and $\hat{x} > A_1^*$ imply $\Lambda_1(\hat{x}) > 0$. Therefore, we get, by (3.12), $U^{(2)}(\hat{x}) < u_{A_2^*}^{\{1\}}(\hat{x}) + u_{A_2^*}^{\{2\}}(\hat{x})$ leading to a contradiction. Because $\hat{x}$ is arbitrary on (A_2^*, ∞), we have the claim. The same contradiction can be derived under Assumption 3.2 because $(\tau_{A_2^*}, \tau_{A_2^*}) \in \widetilde{\mathcal{S}}_2$.

$\square$

The following lemma suggests under $-\infty \leq A_1^* \leq A_2^* \leq \infty$ that the optimal strategy is the simultaneous stopping corresponding to the threshold level $A_{\{1,2\}}^*$, which is the value that makes $\Lambda_{\{1,2\}} \equiv \Lambda_1 + \Lambda_2$ as in (3.8) vanish.

Proposition 3.2. *Suppose* $-\infty \leq A_1^* \leq A_2^* \leq \infty$.

(1) *We have* $A_1^* \leq A_{\{1,2\}}^* \leq A_2^*$.

(2) *It is optimal to stop simultaneously and the value function is given by* $U^{(2)}(x) = u_{A_{\{1,2\}}^*}^{\{1,2\}}(x)$ *under Assumption* 3.1 *and* $\widetilde{U}^{(2)}(x) = u_{A_{\{1,2\}}^*}^{\{1,2\}}(x)$ *under Assumption* 3.2.

Proof. (1) Because both Λ_1 and Λ_2 are increasing, $\Lambda_{\{1,2\}}$ is increasing as well. Because $A_1^* \leq A_2^*$, $\Lambda_1(A_2^*) \geq 0$ and hence $\Lambda_{\{1,2\}}(A_2^*) \geq 0$. Similarly, $\Lambda_{\{1,2\}}(A_1^*) \leq 0$. The increasing property of $\Lambda_{\{1,2\}}$ now shows the claim.

(2) Under Assumption 3.1, for any pair of stopping times $(\tau^{(1)}, \tau^{(2)}) \in \mathcal{S}_2$, because $F_2 = f_2$ as in (3.10) and by the strong Markov property of the Lévy process X,

$$
\begin{aligned}
\mathbb{E}^x &\Big[\sum_{m=1,2} e^{-r\tau^{(m)}} g_m(X_{\tau^{(m)}}) 1_{\{\tau^{(m)} < \infty\}} + \int_0^{\tau^{(1)}} e^{-rt} F_1(X_t) \mathrm{d}t + \int_{\tau^{(1)}}^{\tau^{(2)}} e^{-rt} F_2(X_t) \mathrm{d}t \Big] \\
&= \mathbb{E}^x \Big[e^{-r\tau^{(1)}} g_1(X_{\tau^{(1)}}) 1_{\{\tau^{(1)} < \infty\}} + \int_0^{\tau^{(1)}} e^{-rt} F_1(X_t) \mathrm{d}t \\
&\qquad + 1_{\{\tau^{(1)} < \infty\}} e^{-r\tau^{(1)}} \mathbb{E} \Big[e^{-r(\tau^{(2)} - \tau^{(1)})} g_2(X_{\tau^{(2)}}) 1_{\{\tau^{(2)} < \infty\}} + \int_0^{\tau^{(2)} - \tau^{(1)}} e^{-rt} F_2(X_t) \mathrm{d}t \Big| \mathcal{F}_{\tau^{(1)}} \Big] \Big] \\
&\leq \mathbb{E}^x \Big[e^{-r\tau^{(1)}} g_1(X_{\tau^{(1)}}) 1_{\{\tau^{(1)} < \infty\}} + \int_0^{\tau^{(1)}} e^{-rt} F_1(X_t) \mathrm{d}t + 1_{\{\tau^{(1)} < \infty\}} e^{-r\tau^{(1)}} \sup_{\tau \in \mathcal{S}} u(X_{\tau^{(1)}}, \tau; f_2, g_2) \Big] \\
&= u(x, \tau^{(1)}; F_1, g_1 + u_{A_2^*}^{\{2\}}).
\end{aligned}
$$

(3.15)

Similarly, under Assumption 3.2, (3.15) also holds for any $(\tau^{(1)}, \tau^{(2)}) \in \widetilde{\mathcal{S}}_2$ by replacing $\mathcal{S}$ with $\widetilde{\mathcal{S}}$ in the expectation of the third term. This together with Lemma 3.1, shows $U^{(2)}(x)$ (resp. $\widetilde{U}^{(2)}(x)$) is less than or equal to

$$
\text{(3.16)} \qquad\qquad \sup_{\tau \in \mathcal{S}(A_2^*)} u(x, \tau; F_1, g_1 + u_{A_2^*}^{\{2\}})
$$

under Assumption 3.1 (Assumption 3.2) where $\mathcal{S}(A_2^*)$ is the set of $\tau \in \mathcal{S}$ ($\tau \in \widetilde{\mathcal{S}}$) such that $X_\tau \in (-\infty, A_2^*]$ a.s. on $\{\tau < \infty\}$. Because, for any $\tau \in \mathcal{S}(A_2^*)$, $u_{A_2^*}^{\{2\}}(X_\tau) = g_2(X_\tau)$ a.s. on $\{\tau < \infty\}$ and because $F_1 = f_{\{1,2\}}$ as in (3.10), (3.16) equals

$$
\sup_{\tau \in \mathcal{S}(A_2^*)} u(x, \tau; f_{\{1,2\}}, g_{\{1,2\}}) \leq \sup_{\tau \in \mathcal{S}} u(x, \tau; f_{\{1,2\}}, g_{\{1,2\}}) = u_{A_{\{1,2\}}^*}^{\{1,2\}}(x),
$$

by Corollary 3.1 and because $\mathcal{S}(A_2^*) \subset \mathcal{S}$. Namely, $U^{(2)}(x) \leq u_{A_{\{1,2\}}^*}^{\{1,2\}}(x)$ under Assumption 3.1 and $\widetilde{U}^{(2)}(x) \leq u_{A_{\{1,2\}}^*}^{\{1,2\}}(x)$ under Assumption 3.2. These in fact hold with equality because $u_{A_{\{1,2\}}^*}^{\{1,2\}}(x)$ is attained by $(\tau_{A_{\{1,2\}}^*}, \tau_{A_{\{1,2\}}^*}) \in \widetilde{\mathcal{S}}_2 \subset \mathcal{S}_2$. $\qquad\square$

3.2. **Multiple-stage problem.** We now generalize to the multiple-stage problem and solve (3.1)-(3.2) or equivalently (3.4)-(3.5) with $M \geq 3$. For $1 \leq m \leq M$, let

$$(3.17) \qquad U_m^{(M)}(x) := \sup_{(\tau^{(m)},\ldots,\tau^{(M)}) \in \mathcal{S}_{M-m+1}} \sum_{k=m}^{M} u(x, \tau^{(k)}; f_k, g_k),$$

$$(3.18) \qquad \widetilde{U}_m^{(M)}(x) := \sup_{(\tau^{(m)},\ldots,\tau^{(M)}) \in \widetilde{\mathcal{S}}_{M-m+1}} \sum_{k=m}^{M} u(x, \tau^{(k)}; f_k, g_k).$$

In particular, $U^{(M)} \equiv U_1^{(M)}$ and $\widetilde{U}^{(M)} \equiv \widetilde{U}_1^{(M)}$, and by Corollary 3.1

$$(3.19) \quad U_M^{(M)}(x) = \sup_{\tau \in \mathcal{S}} u(x, \tau; f_M, g_M) = u_{A_M^*}^{\{M\}}(x) \quad \text{and} \quad \widetilde{U}_M^{(M)}(x) = \sup_{\tau \in \widetilde{\mathcal{S}}} u(x, \tau; f_M, g_M) = u_{A_M^*}^{\{M\}}(x),$$

under Assumptions 3.1 and 3.2, respectively. The expressions for $U_{M-1}^{(M)}$ and $\widetilde{U}_{M-1}^{(M)}$ can also be obtained as in the two-stage case.

Given $1 \leq m \leq M$, let us partition $\{m, m+1, \ldots, M\}$ to an $L(m)$ number of (non-empty) disjoint sets $\mathcal{I}_m := \{\mathcal{I}(k; m), 1 \leq k \leq L(m)\}$ such that

$$\{m, m+1, \ldots, M\} = \mathcal{I}(1; m) \cup \cdots \cup \mathcal{I}(L(m); m)$$

where, if $L(m) = 1$, $\mathcal{I}(1; m) = \{m, \ldots, M\}$ and, if $L(m) \geq 2$,

$$\mathcal{I}(1; m) := \{m, \ldots, n_{1,m} - 1\},$$
$$\mathcal{I}(l; m) := \{n_{l-1,m}, \ldots, n_{l,m} - 1\}, \quad 2 \leq l \leq L(m) - 1,$$
$$\mathcal{I}(L(m); m) := \{n_{L(m)-1,m}, \ldots, M\},$$

for some integers $m < n_{1,m} < \cdots < n_{L(m)-1,m} < M$. We consider the strategy such that, if k and l are in the same set, then the k-th and l-th stops occur simultaneously a.s.

We shall show that (3.17) and (3.18), for any $1 \leq m \leq M$, can be solved by a strategy with some partition $\mathcal{I}_m^* := \{\mathcal{I}^*(k; m), 1 \leq k \leq L^*(m)\}$ satisfying

$$A_{\mathcal{I}^*(1;m)}^* > \cdots > A_{\mathcal{I}^*(L^*(m);m)}^*,$$

where $A_{\mathcal{I}}^*$ is defined as in (3.8) for any set $\mathcal{I}$. The corresponding expected value becomes

$$(3.20) \qquad U_{m,\mathcal{I}_m^*}^{(M)}(x) := \sum_{k=1}^{L^*(m)} u(x, \tau_{A_{\mathcal{I}^*(k;m)}^*}; f_{\mathcal{I}^*(k,m)}, g_{\mathcal{I}^*(k;m)}) = \sum_{k=1}^{L^*(m)} u_{A_{\mathcal{I}^*(k;m)}^*}^{\mathcal{I}^*(k;m)}(x),$$

whose strategy is given by for any $m \leq n \leq M$,

$$\tau^{*(n)} = \tau_{A_{\mathcal{I}^*(k;m)}^*} \qquad \text{for the unique } 1 \leq k \leq L^*(m) \text{ such that } n \in \mathcal{I}^*(k; m).$$

We shall show that (3.20) is optimal, i.e. $U_m^{(M)} = U_{m,\mathcal{I}_m^*}^{(M)}$ under Assumption 3.1 and $\widetilde{U}_m^{(M)} = U_{m,\mathcal{I}_m^*}^{(M)}$ under Assumption 3.2 for any $1 \leq m \leq M$. Moreover, $\mathcal{I}_m^*$ can be obtained inductively moving backwards starting from $\mathcal{I}_M^*$ with $L^*(M) = 1$ and $\mathcal{I}^*(1; M) = \{M\}$. For the inductive step, the following algorithm outputs $\mathcal{I}_{m-1}^*$ from $\mathcal{I}_m^*$ for any $2 \leq m \leq M$. By repeating this, we can obtain the partition $\mathcal{I}_1^*$; the resulting $U_{1,\mathcal{I}_1^*}^{(M)}$ as in (3.20) becomes the value function $U^{(M)} = U_1^{(M)}$.

Algorithm $\mathcal{I}^*_{m-1} = \textbf{Update}(\mathcal{I}^*_m, m)$

Step 1: Set $i = 1$.

Step 2: Set

$$\hat{\mathcal{I}} := \begin{cases} \{m-1\}, & i = 1, \\ \{m-1\} \cup \mathcal{I}^*(1; m) \cup \cdots \cup \mathcal{I}^*(i-1; m), & i \geq 2. \end{cases}$$

Step 3: Compute $A^*_{\hat{\mathcal{I}}}$ and

(1) if $i = L^*(m) + 1$, then **stop and return** $\mathcal{I}^*_{m-1} = \{\mathcal{I}^*(1; m-1)\}$ with $L^*(m-1) = 1$ and $\mathcal{I}^*(1; m-1) = \{m-1, \ldots, M\}$;

(2) if $A^*_{\hat{\mathcal{I}}} > A^*_{\mathcal{I}^*(i;m)}$, then **stop and return** $\mathcal{I}^*_{m-1} = \{\mathcal{I}^*(k; m-1), 1 \leq k \leq L^*(m-1)\}$ with $L^*(m-1) = L^*(m) - i + 2$ and

(3.21) $\qquad \mathcal{I}^*(1; m-1) = \hat{\mathcal{I}} \quad \text{and} \quad \mathcal{I}^*(l; m-1) = \mathcal{I}^*(l+i-2; m), \quad 2 \leq l \leq L^*(m-1);$

(3) if $A^*_{\hat{\mathcal{I}}} \leq A^*_{\mathcal{I}^*(i;m)}$, set $i = i + 1$ and go back to **Step 2**.

The role of the algorithm is in words to extend from $n(= M-m+1)$-stage problem to $n+1(= M-m+2)$-stage problem. The idea is similar to what we discussed in the previous section how to extend from a one-stage problem to a two-stage problem. When a new initial stage is added, the corresponding threshold value $A^*_{\hat{\mathcal{I}}}$ is first calculated. Depending on whether its value is higher than that of the subsequent stages or not, simultaneous stoppings may become optimal. For n larger than two, we must solve it recursively by keeping updating the set $\hat{\mathcal{I}}$, or the set of the first (simultaneous) stoppings, as given in this algorithm. If $A^*_{\hat{\mathcal{I}}}$ is low, the strategy of the new initial stage may naturally depend on the strategies of all the subsequent stages. Unlike the extension to the two-stage problem which only needs to take into account the strategy of the stage immediately next, it needs to reflect the strategies of all subsequent stages.

We prove the following under Assumption 3.1 for the optimality (3.1). As is already clear after the detailed discussion on the two-stage case, only a slight modification is needed for (3.2) under Assumption 3.2.

Lemma 3.2. *In view of the algorithm above, suppose Assumption 3.1 and fix $2 \leq m \leq M$. Given that $\mathcal{I}^*_m$ satisfies, for every $1 \leq l \leq L^*(m)$,*

(3.22) $$U^{(M)}_{\min \mathcal{I}^*(l;m)}(x) = \sum_{k=l}^{L^*(m)} u^{\mathcal{I}^*(k;m)}_{A^*_{\mathcal{I}^*(k;m)}}(x),$$

and is used as an input in the algorithm. Then, we have the following.

(1) *At the end of **Step 2**, if $1 \leq i \leq L^*(m)$,*

(3.23) $$U^{(M)}_{m-1}(x) \leq \sup_{\tau \in \mathcal{S}} \left[u(x, \tau; f_{\hat{\mathcal{I}}}, g_{\hat{\mathcal{I}}}) + \sum_{k=i}^{L^*(m)} u(x, \vartheta_\tau(A^*_{\mathcal{I}^*(k;m)}); f_{\mathcal{I}^*(k;m)}, g_{\mathcal{I}^*(k;m)}) \right]$$

where $\vartheta_\nu(A) := \nu + \tau_A \circ \theta_\nu$ for any $\nu \in \mathcal{S}$ and $A \in \mathbb{R}$ with the time-shift operator θ_t, and if $i = L^*(m) + 1$

$$U^{(M)}_{m-1}(x) = \sup_{\tau \in \mathcal{S}} u(x, \tau; f_{\hat{\mathcal{I}}}, g_{\hat{\mathcal{I}}}) = u^{\{m-1,\ldots,M\}}_{A^*_{\{m-1,\ldots,M\}}}(x).$$

(2) *Let $\mathcal{I}_{m-1}^*$ be produced by the algorithm. For any $1 \le l \le L^*(m-1)$,*

$$
(3.24) \qquad U_{\min \mathcal{I}^*(l;m-1)}^{(M)}(x) = \sum_{k=l}^{L^*(m-1)} u_{A_{\mathcal{I}^*(k;m-1)}^*}^{\mathcal{I}^*(k;m-1)}(x).
$$

Proof. (1) We shall show by mathematical induction.

(Base-step) Suppose $i = 1$. By our assumption (3.22) and by an argument similar to (3.15),

$$
U_{m-1}^{(M)}(x) \le \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \Big[e^{-r\tau} g_{m-1}(X_\tau) 1_{\{\tau < \infty\}} + \int_0^\tau e^{-rt} F_{m-1}(X_t) \mathrm{d}t + e^{-r\tau} U_m^{(M)}(X_\tau) 1_{\{\tau < \infty\}} \Big]
$$

$$
= \sup_{\tau \in \mathcal{S}} \Big[u(x, \tau; f_{m-1}, g_{m-1}) + \sum_{k=1}^{L^*(m)} u(x, \vartheta_\tau(A_{\mathcal{I}^*(k;m)}^*); f_{\mathcal{I}^*(k;m)}, g_{\mathcal{I}^*(k;m)}) \Big].
$$

Now (3.23) holds because $\hat{\mathcal{I}} = \{m-1\}$ for $i = 1$.

(Inductive-step) Now we assume (3.23) for $i = j \le L^*(m) - 1$, i.e., $\hat{\mathcal{I}} = \{m-1\} \cup \mathcal{I}^*(1;m) \cup \cdots \cup \mathcal{I}^*(j-1;m)$ and

$$
(3.25) \qquad U_{m-1}^{(M)}(x) \le \sup_{\tau \in \mathcal{S}} \Big[u(x, \tau; f_{\hat{\mathcal{I}}}, g_{\hat{\mathcal{I}}}) + \sum_{k=j}^{L^*(m)} u(x, \vartheta_\tau(A_{\mathcal{I}^*(k;m)}^*); f_{\mathcal{I}^*(k;m)}, g_{\mathcal{I}^*(k;m)}) \Big],
$$

and show that it will hold for $i = j + 1$.

Because when $A_{\hat{\mathcal{I}}}^* > A_{\mathcal{I}^*(j;m)}^*$ the algorithm stops at j and never returns to **Step 2**, we suppose here that $A_{\hat{\mathcal{I}}}^* \le A_{\mathcal{I}^*(j;m)}^*$. In view of the right-hand side of (3.25), if there exists some $\hat{x} > A_{\mathcal{I}^*(j;m)}^*$ at which it is optimal to stop, then the value function becomes $g_{\hat{\mathcal{I}}}(\hat{x}) + U_{\min \mathcal{I}^*(j;m)}^{(M)}(\hat{x})$ by our assumption (3.22). Using the same reasoning as in Lemma 3.1, this is in fact smaller than $u_{A_{\mathcal{I}^*(j;m)}^*}^{\hat{\mathcal{I}}}(\hat{x}) + U_{\min \mathcal{I}^*(j;m)}^{(M)}(\hat{x})$. Hence is is never optimal to stop on $(A_{\mathcal{I}^*(j;m)}^*, \infty)$ for the optimization problem on the right-hand side of (3.25) (see also the proof of Proposition 3.2).

Now let $\mathcal{S}(A_{\mathcal{I}^*(j;m)}^*)$ be the set of all stopping times at which $X \in (-\infty, A_{\mathcal{I}^*(j;m)}^*]$ a.s. For all $\tau \in \mathcal{S}(A_{\mathcal{I}^*(j;m)}^*)$, we have $\tau = \vartheta_\tau(A_{\mathcal{I}^*(j;m)}^*)$ a.s. and hence $u(x, \vartheta_\tau(A_{\mathcal{I}^*(j;m)}^*); f_{\mathcal{I}^*(j;m)}, g_{\mathcal{I}^*(j;m)}) = u(x, \tau; f_{\mathcal{I}^*(j;m)}, g_{\mathcal{I}^*(j;m)})$. Therefore (3.25) implies

$$
U_{m-1}^{(M)}(x) \le \sup_{\tau \in \mathcal{S}(A_{\mathcal{I}^*(j;m)}^*)} \Big[u(x, \tau; f_{\hat{\mathcal{I}} \cup \mathcal{I}^*(j;m)}, g_{\hat{\mathcal{I}} \cup \mathcal{I}^*(j;m)}) + \sum_{k=j+1}^{L^*(m)} u(x, \vartheta_\tau(A_{\mathcal{I}^*(k;m)}^*); f_{\mathcal{I}^*(k;m)}, g_{\mathcal{I}^*(k;m)}) \Big]
$$

$$
\le \sup_{\tau \in \mathcal{S}} \Big[u(x, \tau; f_{\hat{\mathcal{I}} \cup \mathcal{I}^*(j;m)}, g_{\hat{\mathcal{I}} \cup \mathcal{I}^*(j;m)}) + \sum_{k=j+1}^{L^*(m)} u(x, \vartheta_\tau(A_{\mathcal{I}^*(k;m)}^*); f_{\mathcal{I}^*(k;m)}, g_{\mathcal{I}^*(k;m)}) \Big].
$$

Hence, (3.23) holds for $i = j + 1$, as desired. This proves (1) by mathematical induction.

(2) When the algorithm stops, it is either (i) $i = L^*(m) + 1$ at **Step 3** (1) or (ii) $A_{\hat{\mathcal{I}}}^* > A_{\mathcal{I}^*(i;m)}^*$ at **Step 3** (2).

(i) Suppose $i = L^*(m) + 1$. In this case, $\hat{\mathcal{I}} = \{m-1, \ldots, M\}$ and, by (3.23),

$$
U_{\min \mathcal{I}^*(1;m-1)}^{(M)}(x) = U_{m-1}^{(M)}(x) \le \sup_{\tau \in \mathcal{S}} u(x, \tau; f_{\{m-1,\ldots,M\}}, g_{\{m-1,\ldots,M\}}) = u_{A_{\{m-1,\ldots,M\}}^*}^{\{m-1,\ldots,M\}}(x),
$$

which in fact holds by equality because the right-hand side is attained by $(\tau_{A_{\{m-1,\ldots,M\}}^*}, \ldots, \tau_{A_{\{m-1,\ldots,M\}}^*}) \in \widetilde{\mathcal{S}}_{M-m+2} \subset \mathcal{S}_{M-m+2}$.

(ii) Suppose the algorithm exits at i with $A_{\hat{\mathcal{I}}}^* > A_{\mathcal{I}^*(i;m)}^*$. By (3.23), we have

$$U_{m-1}^{(M)}(x) \le \sup_{\tau \in \mathcal{S}} u(x, \tau; f_{\hat{\mathcal{I}}}, g_{\hat{\mathcal{I}}}) + \sup_{\tau \in \mathcal{S}} \sum_{k=i}^{L^*(m)} u(x, \vartheta_\tau(A_{\mathcal{I}^*(k;m)}^*); f_{\mathcal{I}^*(k;m)}, g_{\mathcal{I}^*(k,m)}).$$

Regarding the second supremum of the right-hand side, the strategy $\{\tau^{(l)}; \min \mathcal{I}^*(i, m) \le l \le M\}$, defined by $\tau^{(l)} = \vartheta_\tau(A_{\mathcal{I}^*(k;m)}^*)$ for the unique $i \le k \le L^*(m)$ such that $l \in \mathcal{I}^*(k; m)$, is feasible (or in $\mathcal{S}_{M+1-\min \mathcal{I}^*(i,m)}$) for any stopping time $\tau \in \mathcal{S}$ and therefore

$$\sup_{\tau \in \mathcal{S}} \sum_{k=i}^{L^*(m)} u(x, \vartheta_\tau(A_{\mathcal{I}^*(k;m)}^*); f_{\mathcal{I}^*(k;m)}, g_{\mathcal{I}^*(k,m)})$$

$$\le \sup_{(\tau^{(\min \mathcal{I}^*(i,m))}, \dots, \tau^{(M)}) \in \mathcal{S}_{M+1-\min \mathcal{I}^*(i,m)}} \sum_{k=\min \mathcal{I}^*(i;m)}^{M} u(x, \tau^{(k)}; f_k, g_k) = U_{\min \mathcal{I}^*(i;m)}^{(M)}(x).$$

Hence, we obtain a bound $U_{m-1}^{(M)}(x) \le u_{A_{\hat{\mathcal{I}}}^*}^{\hat{\mathcal{I}}}(x) + U_{\min \mathcal{I}^*(i;m)}^{(M)}(x)$. This together with (3.21) and (3.22) shows

$$U_{m-1}^{(M)}(x) \le u_{A_{\hat{\mathcal{I}}}^*}^{\hat{\mathcal{I}}}(x) + \sum_{k=i}^{L^*(m)} u_{A_{\mathcal{I}^*(k;m)}^*}^{\mathcal{I}^*(k;m)}(x) = \sum_{k=1}^{L^*(m-1)} u_{A_{\mathcal{I}^*(k;m-1)}^*}^{\mathcal{I}^*(k;m-1)}(x).$$

This holds by equality because the left-hand side is attained by a feasible strategy defined by $\mathcal{I}_{m-1}^*$. This shows (3.24) for case $l = 1$.

On the other hand, for any $2 \le l \le L^*(m-1)$, by (3.21) and (3.22),

$$U_{\min \mathcal{I}^*(l;m-1)}^{(M)}(x) = U_{\min \mathcal{I}^*(l+i-2;m)}^{(M)}(x) = \sum_{k=l+i-2}^{L^*(m)} u_{A_{\mathcal{I}^*(k;m)}^*}^{\mathcal{I}^*(k;m)}(x) = \sum_{k=l}^{L^*(m-1)} u_{A_{\mathcal{I}^*(k;m-1)}^*}^{\mathcal{I}^*(k;m-1)}(x),$$

which guarantees (3.24), as desired.

$\square$

Using Lemma 3.2 as an inductive step, the main theorem is immediate. Indeed, (3.22) holds trivially for M by Corollary 3.1. By applying the algorithm $M - 1$ times, we can obtain (3.1) for $M - 1, M - 2, \dots, 1$.

Theorem 3.1. *Let $\{\mathcal{I}_m^*; 1 \le m \le M\}$ be produced by the algorithm.*

(1) *Under Assumption 3.1, for every $1 \le m \le M$ and $1 \le i \le L^*(m)$,*

$$U_{\min \mathcal{I}^*(i;m)}^{(M)}(x) = \sum_{k=i}^{L^*(m)} u_{A_{\mathcal{I}^*(k;m)}^*}^{\mathcal{I}^*(k;m)}(x).$$

In particular,

(3.26)
$$U^{(M)}(x) \equiv U_{\min \mathcal{I}^*(1;1)}^{(M)}(x) = \sum_{k=1}^{L^*(1)} u_{A_{\mathcal{I}^*(k;1)}^*}^{\mathcal{I}^*(k;1)}(x).$$

(2) *Under Assumption 3.2, for every $1 \leq m \leq M$ and $1 \leq i \leq L^*(m)$,*

$$\widetilde{U}_{\min \mathcal{I}^*(i;m)}^{(M)}(x) = \sum_{k=i}^{L^*(m)} u_{A_{\mathcal{I}^*(k;m)}^*}^{\mathcal{I}^*(k;m)}(x).$$

In particular,

(3.27)
$$\widetilde{U}^{(M)}(x) \equiv \widetilde{U}_{\min \mathcal{I}^*(1;1)}^{(M)}(x) = \sum_{k=1}^{L^*(1)} u_{A_{\mathcal{I}^*(k;1)}^*}^{\mathcal{I}^*(k;1)}(x).$$

4. HYPEREXPONENTIAL JUMP DIFFUSION CASE AND NUMERICAL EXAMPLES

In this section, we consider spectrally negative Lévy processes with i.i.d. hyperexponential jumps and provide numerical examples. If a Lévy measure has a completely monotone density, it can be approximated arbitrarily closely by that of a compound Poisson process with hyperexponential jumps (see, e.g., [18, 19]). In a related work, Asmussen et al. [3] approximate the Lévy density of the CGMY process by a hyperexponential density. Herein, we will use the explicit expression of the scale function obtained by [18].

4.1. **Spectrally Negative Lévy Processes with Hyperexponential Jumps.** Let X be a spectrally negative Lévy process of the form

(4.1)
$$X_t - X_0 = \mu t + \sigma B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \leq t < \infty.$$

Here $B = \{B_t; t \geq 0\}$ is a standard Brownian motion, $N = \{N_t; t \geq 0\}$ is a Poisson process with arrival rate λ, and $Z = \{Z_n; n = 1, 2, \ldots\}$ is an i.i.d. sequence of hyperexponential random variables with density function

$$h(z) := \sum_{j=1}^{J} p_j \eta_j e^{-\eta_j z}, \quad z > 0,$$

for some $p_1 + \cdots + p_J = 1$ and $0 < \eta_1 < \cdots < \eta_J < \infty$. Its Laplace exponent (2.1) is given by

$$\psi(s) = \mu s + \frac{1}{2}\sigma^2 s^2 - \lambda \sum_{j=1}^{J} p_j \frac{s}{\eta_j + s}, \quad s \in \mathbb{R}.$$

In particular, this reduces to a Brownian motion when $J = 0$ and to (spectrally negative) exponential jump diffusion when $J = 1$.

For our examples, we assume $\sigma > 0$; see [18] for the case $\sigma = 0$. In this case, there are $J+1$ *negative* solutions to the equation $\psi(s) = r$ and their absolute values $\{\xi_{i,r}; i = 1, \ldots, J+1\}$ satisfy the interlacing condition: $0 < \xi_{1,r} < \eta_1 < \xi_{2,r} < \cdots < \eta_J < \xi_{J+1,r} < \infty$. For this process, the scale functions are given by

(4.2)
$$W^{(r)}(x) = \sum_{i=1}^{J+1} C_i \left[e^{\Phi_r x} - e^{-\xi_{i,r} x} \right] \quad \text{and} \quad Z^{(r)}(x) = 1 + r \sum_{i=1}^{J+1} C_i \left[\frac{1}{\Phi_r} \left(e^{\Phi_r x} - 1 \right) + \frac{1}{\xi_{i,r}} \left(e^{-\xi_{i,r} x} - 1 \right) \right]$$

for every $x \geq 0$ where

$$C_i := A_{i,r} \frac{2}{\sigma^2 \sum_{k=1}^{J+1} A_{k,r} \xi_{k,r}} \left(\frac{\xi_{i,r}}{\Phi_r + \xi_{i,r}} \right) \quad \text{with} \quad A_{i,r} := \frac{\prod_{j \in \{1,\ldots,J\}} \left(1 - \frac{\xi_{i,r}}{\eta_j} \right)}{\prod_{l \in \{1,\ldots,J+1\} \setminus \{i\}} \left(1 - \frac{\xi_{i,r}}{\xi_{l,r}} \right)}, \quad 1 \leq i \leq J+1.$$

With $\Phi_r^{(c)} := \Phi_r - c$ and $\xi_{i,r}^{(c)} := \xi_{i,r} + c$, $1 \leq i \leq J+1$, for any $c \geq 0$, we have by (2.6) for any $x \geq 0$

(4.3)
$$W_c^{(r-\psi(c))}(x) = \sum_{i=1}^{J+1} C_i \left[e^{\Phi_r^{(c)} x} - e^{-\xi_{i,r}^{(c)} x} \right],$$

$$Z_c^{(r-\psi(c))}(x) = 1 + (r - \psi(c)) \sum_{i=1}^{J+1} C_i \left[\frac{1}{\Phi_r^{(c)}} (e^{\Phi_r^{(c)} x} - 1) + \frac{1}{\xi_{i,r}^{(c)}} (e^{-\xi_{i,r}^{(c)} x} - 1) \right].$$

Thanks to their forms as sums of exponential functions, the value function we obtain below admits closed forms.

4.2. **Numerical results on the one-stage problem.**
We first consider the one-stage problem as studied in Section 2. In our numerical examples, we consider two examples for g:

(a) *mixture of exponential functions:* $g^{(exp)}$ as in (2.20);
(b) *linear function:* $g^{(lin)}(x) := -\alpha x + \beta$, $x \in \mathbb{R}$, for some $\alpha > 0$ and $\beta \in \mathbb{R}$.

It is clear that $g^{(lin)}$ satisfies Assumption 2.3 (2). Regarding f, we consider the following three examples:

(i) *simple function:* $f^{(sim)}(y) := \sum_{-\infty < n < \infty} f^{(n)} 1_{I_n}(y)$ for some constants $\cdots < f^{(-2)} < f^{(-1)} < f^{(0)} < f^{(1)} < f^{(2)} < \cdots$ such that $-\infty < \lim_{n \downarrow -\infty} f^{(n)} \leq \lim_{n \uparrow \infty} f^{(n)} < \infty$ and subdivisions $I_n := (l_n, l_{n+1}]$ of $\mathbb{R}$;

(ii) *linear function with a lower bound:* $f^{(lin)}(y) := b_1[(y + b_2) \vee b_3]$ for some $b_1 > 0$ and $b_2, b_3 \in \mathbb{R}$;

(iii) *exponential function with an upper bound:* $f^{(exp)}(y) := e^{(Ly) \wedge B}$ for some $L > 0$ and $B \in \mathbb{R}$.

These satisfy Assumption 2.3 (1) and in particular (ii) and (iii) satisfy Assumption 2.2 (1). Hence Proposition 2.2 holds (or $\widetilde{u} = u_{A^*}$) for any choice and in particular Proposition 2.1 holds (or $u = \widetilde{u} = u_{A^*}$) for (a) with (ii) or (iii).

In order to implement the optimal strategy, we first obtain A^* using (2.17) (or (2.22) for (a)) and then compute the value function via (2.25) (or (2.23) for (a)). Toward this end, we need to compute $\rho_{g^{(lin)},A}^{(r)}$ and $\varphi_{g^{(lin)},A}^{(r)}(x)$ and $\Psi_f(A)$ and $\Theta_f(x; A)$ for (i)-(iii). The proofs of the following two lemmas are tedious but straightforward and hence omitted.

Lemma 4.1. *(1) For every $A \in \mathbb{R}$, we have*

$$\rho_{g^{(lin)},A}^{(r)} = -\frac{\alpha \lambda}{\Phi_r^2} \left[1 - \sum_{j=1}^{J} p_j \left(\frac{\eta_j}{\eta_j + \Phi_r} + \frac{\Phi_r}{\eta_j} \right) \right].$$

(2) For all $x \geq A$,

$$\varphi_{g^{(lin)},A}^{(r)}(x) = \alpha \lambda \sum_{i=1}^{J+1} C_i \sum_{j=1}^{J} p_j \left[\frac{1}{\eta_j(\eta_j + \Phi_r)} (e^{\Phi_r(x-A)} - e^{-\eta_j(x-A)}) - \frac{1}{\eta_j(\eta_j - \xi_{i,r})} (e^{-\xi_{i,r}(x-A)} - e^{-\eta_j(x-A)}) \right].$$

Define, for any $s < t$ and $k \geq 0$,

$$w_{x,A}^{(k)}(s,t) := \sum_{i=1}^{J+1} C_i \left[\frac{1}{\Phi_r^{(k)}} \left(e^{\Phi_r^{(k)}(x-s\vee A)_+} - e^{\Phi_r^{(k)}(x-t\vee A)_+} \right) + \frac{1}{\xi_{i,r}^{(k)}} \left(e^{-\xi_{i,r}^{(k)}(x-s\vee A)_+} - e^{-\xi_{i,r}^{(k)}(x-t\vee A)_+} \right) \right].$$

This vanishes whenever $[s,t] \cap [A,x] = \emptyset$. Also let for any $s < t$ and $a \in \mathbb{R}$, $\gamma^{(a)}(s,t) := \int_s^t y e^{ay} \mathrm{d}y = \left(\frac{t}{a} - \frac{1}{a^2} \right) e^{at} - \left(\frac{s}{a} - \frac{1}{a^2} \right) e^{as}$.

Lemma 4.2. *(1) For every $A \in \mathbb{R}$,*

$$\Psi_{f^{(sim)}}(A) = \frac{1}{\Phi_r} \sum_{-\infty < n < \infty} f^{(n)} \left[e^{-\Phi_r(l_n - A)_+} - e^{-\Phi_r(l_{n+1}-A)_+} \right],$$

$$\Psi_{f^{(lin)}}(A) = b_1 \left[\frac{b_3}{\Phi_r}(1 - e^{-\Phi_r[b_3-(b_2+A)]_+}) + \frac{b_2 + A}{\Phi_r} e^{-\Phi_r[b_3-(b_2+A)]_+} + e^{-\Phi_r[b_3-(b_2+A)]_+} \left[\frac{[b_3 - (b_2+A)]_+}{\Phi_r} + \frac{1}{\Phi_r^2} \right] \right],$$

$$\Psi_{f^{(exp)}}(A) = \frac{e^B}{\Phi_r} e^{-\Phi_r(\frac{B}{L}-A)_+} + \frac{e^{LA}}{\Phi_r^{(L)}} \left[1 - e^{-\Phi_r^{(L)}(\frac{B}{L}-A)_+} \right].$$

In particular, for any sufficiently small b_3, $\Psi_{f^{(lin)}}(A) = b_1 \left[\frac{b_2+A}{\Phi_r} + \frac{1}{\Phi_r^2} \right]$.
(2) For every $x, A \in \mathbb{R}$,

$$\Theta_{f^{(sim)}}(x;A) = \sum_{-\infty < n < \infty} f^{(n)} w_{x,A}^{(0)}(l_n, l_{n+1}),$$

$$\Theta_{f^{(lin)}}(x;A) = b_1 \left[b_3 w_{x,A}^{(0)}(A, (b_3 - b_2) \vee A \wedge x) + b_2 w_{x,A}^{(0)}((b_3 - b_2) \vee A \wedge x, x) \right]$$
$$+ b_1 \sum_{i=1}^{J+1} C_i [e^{\Phi_r x} \gamma^{(-\Phi_r)}((b_3 - b_2) \vee A \wedge x, x) - e^{-\xi_{i,r}x} \gamma^{(\xi_{i,r})}((b_3 - b_2) \vee A \wedge x, x)],$$

$$\Theta_{f^{(exp)}}(x;A) = w_{x,A}^{(0)}(A \vee (B/L) \wedge x, x) + w_{x,A}^{(L)}(A, A \vee (B/L) \wedge x).$$

We are now ready to show our numerical results for the one-stage problem. Throughout the results given below, we use the common Lévy process of the form (4.1) with $\mu = 1$, $\sigma = 0.2$, $J = 2$, $p = [0.1, 0.9]$, $\eta = [1, 20]$ and $\lambda = 5$. Namely, this is a hyperexponential Lévy process with two kinds of negative jumps: frequent and small jumps and infrequent and large jumps. For g and f, we consider any combination of the following:

   (a) $g = g^{(exp)}$ with $a = [0.1, 0.2, 0.3, 0.4]$ and $c = [4, 3, 2, 1]$;
   (b) $f = g^{(lin)}$ with $\alpha = 1$ and $\beta = 0$;

and

   (i) $f = \gamma f^{(sim)}$ with $I_1 = (-\infty, 0)$, $I_2 = [0, \infty)$, $f^{(1)} = -10$ and $f^{(2)} = 10$;
   (ii) $f = \gamma f^{(lin)}$ with $b_1 = 1$ and $b_2 = 0$ with sufficiently small b_3;
   (iii) $f = \gamma f^{(exp)}$ with $L = B = 1$;

for the weight parameter $\gamma = 0, 0.05, 0.1$.

The results for (a) $g = g^{(exp)}$ and (b) $g = g^{(lin)}$ are graphically shown in Figures 1 and 2, respectively. In each figure, we plot the function $\Lambda(\cdot)$ as in (2.17) and the value function u_{A^*} for each choice of f. As can be confirmed, the function $\Lambda(\cdot)$ is indeed monotonically increasing and hence the unique root A^* of $\Lambda(A) = 0$ can be obtained

easily by the bisection method. Using these optimal threshold levels, the value functions are computed via (2.23) and (2.25) using Lemmas 4.1 and 4.2.

We see that the value functions are differentiable even at the optimal threshold levels A^* and this confirms the smooth fit as in Remark 2.4 because X is of unbounded variation with $\sigma > 0$. In particular, for the case $\gamma = 0$ with no running reward ($f \equiv 0$), because the stopping value decreases to $-\infty$ as x increases, the value function converges to zero as x increases. For $f^{(sim)}$ with a discontinuity at zero, the value function turns out to have a kink at zero although it still appears to be differentiable. While these computations require bisection-type methods to obtain $\xi_{i,r}$'s, Φ_r and A^*, these can be solved instantaneously.

In order to verify that these are indeed optimal, we focus on the case $\gamma = 0.05$ and plot in Figure 3 the value function u_{A^*} in comparison with the expected values of "perturbed" strategies $u_A(\cdot)$ for $A = A^* - 2, A^* - 1, A^* + 1, A^* + 2$. Notice that these can be computed as the sum of Γ's as in (2.14). For any choice of A, it is easy to see that u_A is continuous as in Remark 2.4 but fails to be differentiable at $A \neq A^*$. We can confirm in all six cases that u_{A^*} indeed dominates u_A for $A \neq A^*$ uniformly in $x \in \mathbb{R}$. This numerically verifies Propositions 2.1 and 2.2.

4.3. **Numerical results on the multiple-stage problem.** We now move onto the multiple-stage problem. We assume $M = 3$ for brevity and use for f and g the functions (a)-(b) and (i)-(iii) defined for the one-stage problem.

We first verify (3.26) or the optimality over $\mathcal{S}_3$ for the case, for each $1 \leq m \leq 3$, (a) $g_m = g^{(exp)}$ for some $a_m = (a_{mi})_{1 \leq i \leq N_m}$ and $c_m = (c_{mi})_{1 \leq i \leq N_m}$ with $N_m = 4$, and either (ii) $f_m = \gamma_m f^{(lin)}$ with $b_1 = 1$ and $b_2 = 0$ with sufficiently small b_3 or (iii) $f_m = \gamma_m f^{(exp)}$ with $L = B = 1$ for some $\gamma_m > 0$.

We conduct a number of experiments for various values of $\{a_m, c_m, \gamma_m, 1 \leq m \leq 3\}$. By using the algorithm given in Subsection 3.2, the optimal threshold levels $A^* = (A^{*(1)}, A^{*(2)}, A^{*(3)})$ takes values among $\{A_1^*, A_2^*, A_3^*, A_{\{1,2\}}^*, A_{\{2,3\}}^*, A_{\{1,2,3\}}^*\}$ and satisfy one of the following four cases:

Case 1: $A^{*(1)} = A^{*(2)} = A^{*(3)}$;
Case 2: $A^{*(1)} > A^{*(2)} = A^{*(3)}$;
Case 3: $A^{*(1)} = A^{*(2)} > A^{*(3)}$;
Case 4: $A^{*(1)} > A^{*(2)} > A^{*(3)}$.

Here we use a random random number generator to sample a_m, c_m and γ_m for each $1 \leq m \leq 3$ and also the choice of $f^{(lin)}/f^{(exp)}$ until we attain each of Cases 1 to 4. The generated parameters and the corresponding threshold levels are summarized in Table 1. In order to validate the optimality of the strategy $(\tau_{A^{*(1)}}, \tau_{A^{*(2)}}, \tau_{A^{*(3)}})$, we compare in Figure 4 the value function with those of perturbed strategies $(\tau_{\widetilde{A}_k^{(1)}}, \tau_{\widetilde{A}_k^{(2)}}, \tau_{\widetilde{A}_k^{(3)}})$, $1 \leq k \leq 6$, where

$$\widetilde{A}_k := (A^{*(1)}, A^{*(2)}, A^{*(3)}) + \delta_k$$

with $\delta_1 := (1, 0, 0)$, $\delta_2 := (1, 1, 0)$, $\delta_3 := (1, 1, 1)$, $\delta_4 := (0, 0, -1)$, $\delta_5 := (0, -1, -1)$ and $\delta_6 := (-1, -1, -1)$. It is clear that $(\tau_{\widetilde{A}_k^{(1)}}, \tau_{\widetilde{A}_k^{(2)}}, \tau_{\widetilde{A}_k^{(3)}}) \in \widetilde{\mathcal{S}}_3 \subset \mathcal{S}_3$ because $\widetilde{A}_k^{(1)} \geq \widetilde{A}_k^{(2)} \geq \widetilde{A}_k^{(3)}$ by construction. Figure 4 suggests in all cases that the value obtained by $(\tau_{A^{*(1)}}, \tau_{A^{*(2)}}, \tau_{A^{*(3)}})$ dominates uniformly over x those obtained by the perturbed strategies. These results are indeed consistent with our main theoretical results as in (3.26). In view of Figure 4, we also observe that there are up to three kinks (at A^*) in the value function although these still appear to be differentiable. This is again due to smooth fit as in Remark 2.4. The perturbed strategies on the other hand fail to be differentiable while they are still continuous.
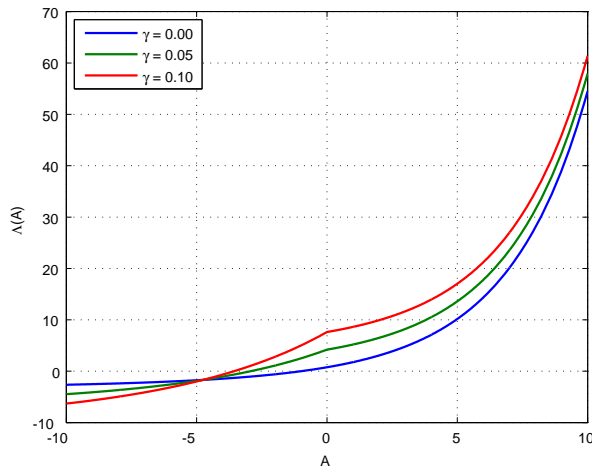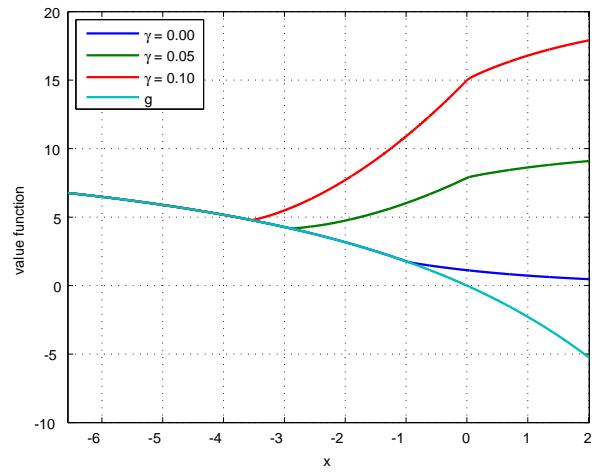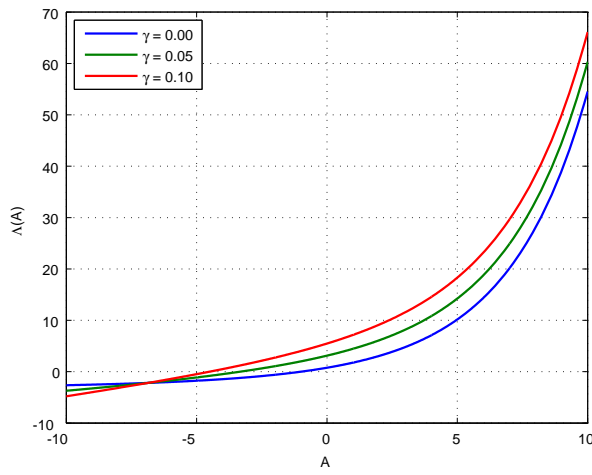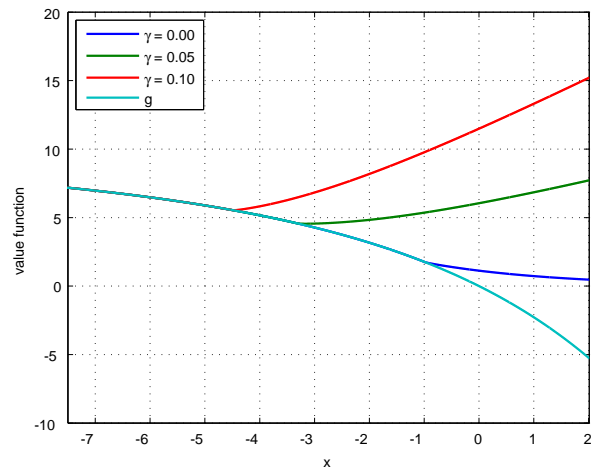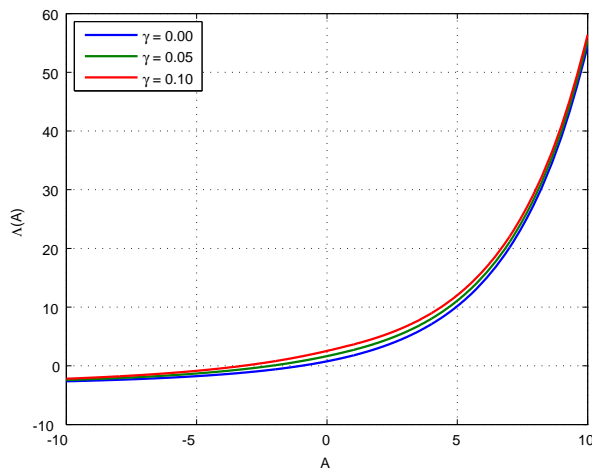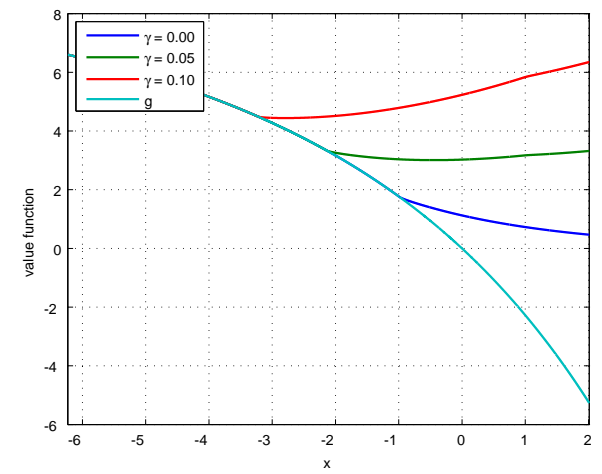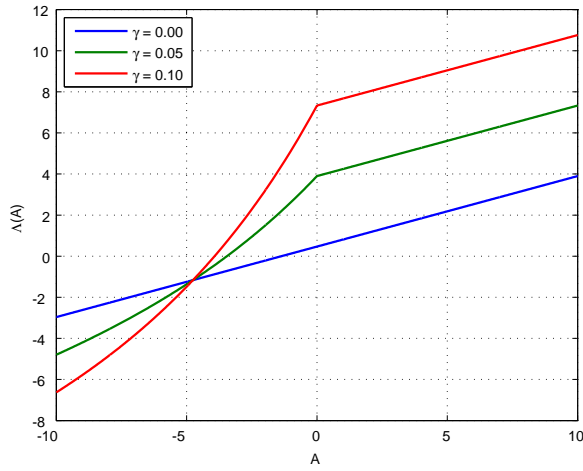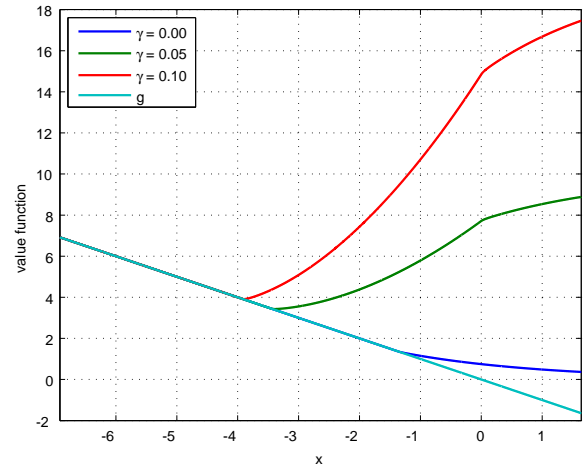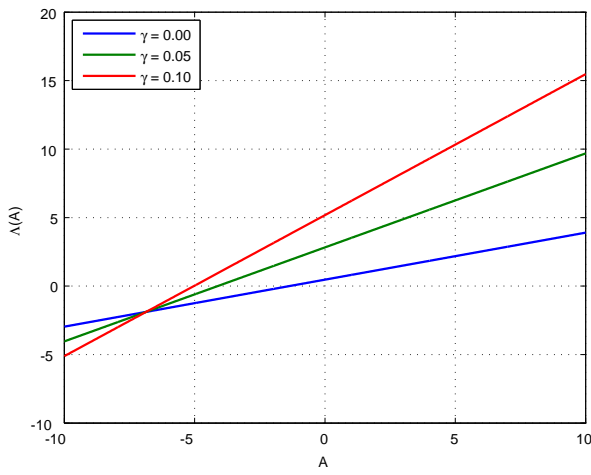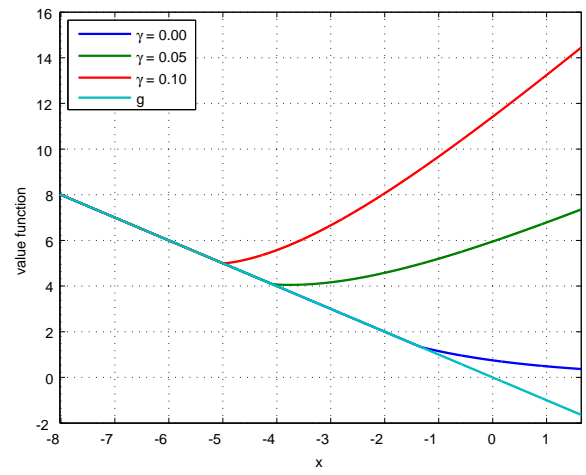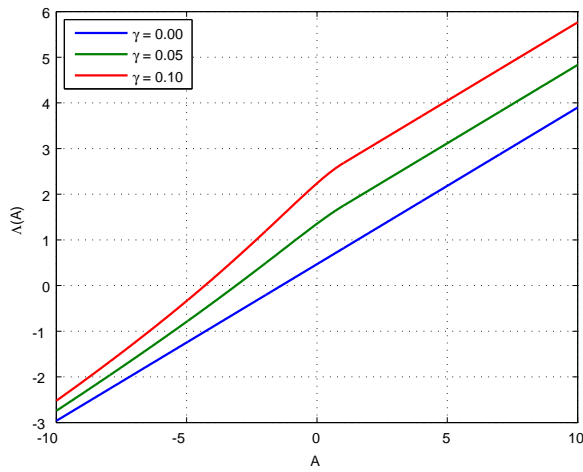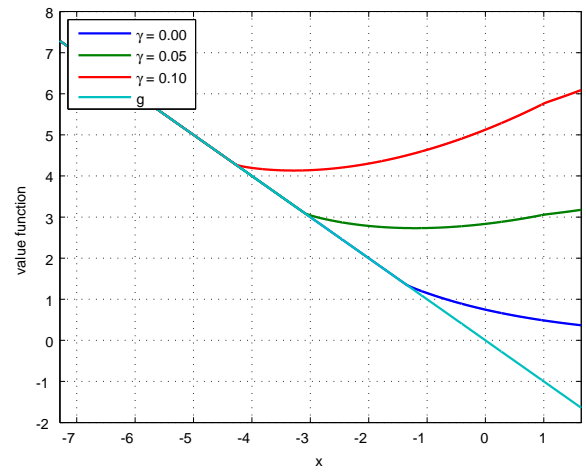
FIGURE 1. Plots of Λ and the value function u_{A^*} when $g = g^{(exp)}$.

FIGURE 2. Plots of Λ and the value function u_{A^*} when $g = g^{(lin)}$.
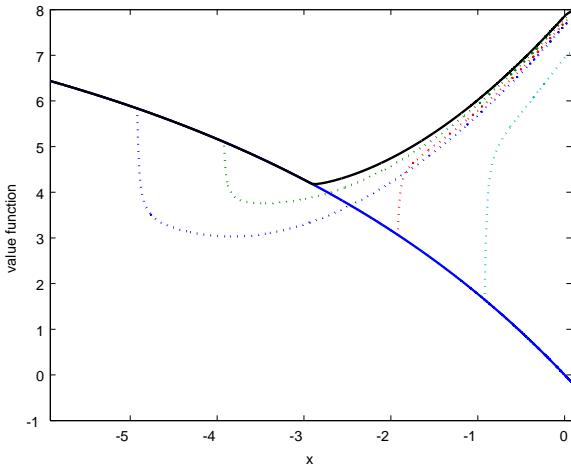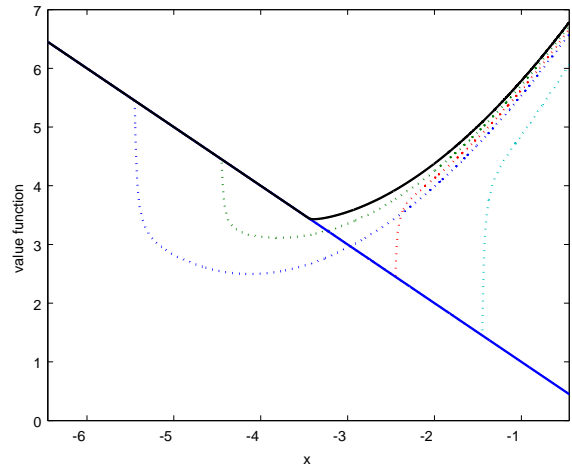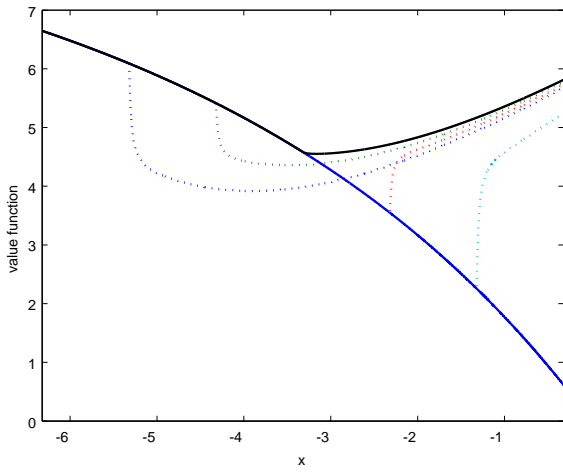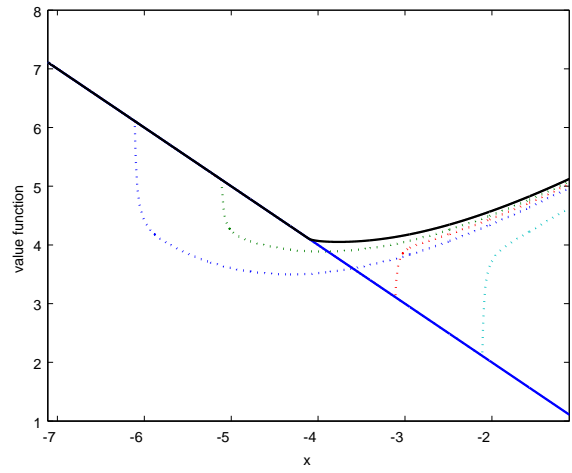
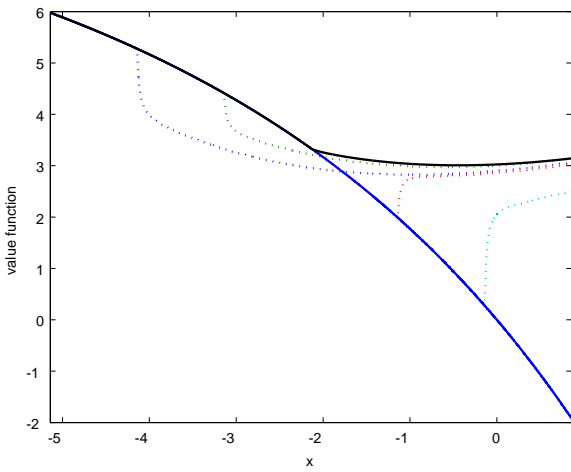(a-i) $g^{(exp)}$ with $f^{(sim)}$
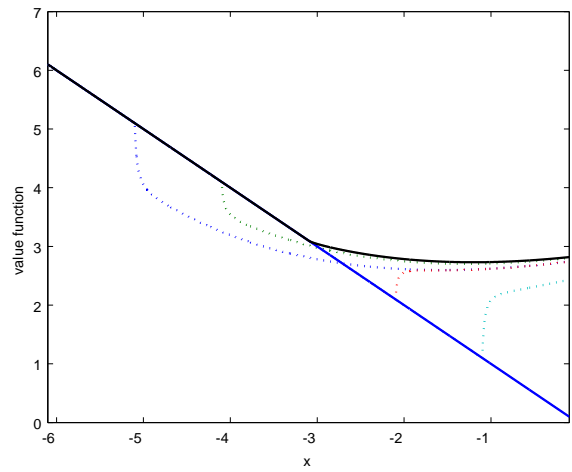
(b-i) $g^{(lin)}$ with $f^{(sim)}$

(a-ii) $g^{(exp)}$ with $f^{(lin)}$

(b-ii) $g^{(lin)}$ with $f^{(lin)}$

(a-iii) $g^{(exp)}$ with $f^{(exp)}$

(b-iii) $g^{(lin)}$ with $f^{(exp)}$

FIGURE 3. Verification of optimality: the value function u_{A^*} (solid black) and the stopping value g (solid blue) in comparison with $u_A(\cdot)$ for $A = A^* - 2, A^* - 1, A^* + 1, A^* + 2$ (dotted).

	f_1	f_2	f_3	γ_1	γ_2	γ_3
Case 1	lin	exp	lin	0.2287	0.0642	0.7673
Case 2	lin	exp	exp	0.2834	0.8962	0.8266
Case 3	lin	exp	lin	0.8131	0.3833	0.6173
Case 4	exp	exp	exp	0.4139	0.3091	0.2638

	a_1	a_2	a_3
Case 1	(0.34, 0.21, 0.16, 0.43)	(0.36, 0.20, 0.41, 0.25)	(0.32, 0.41, 0.39, 0.32)
Case 2	(0.20, 0.42, 0.16, 0.44)	(0.25, 0.30, 0.23, 0.36)	(0.35, 0.29, 0.36, 0.01)
Case 3	(0.29, 0.12, 0.40, 0.27)	(0.27, 0.23, 0.49, 0.04)	(0.14, 0.11, 0.01, 0.40)
Case 4	(0.38, 0.40, 0.41, 0.25)	(0.50, 0.10, 0.21, 0.40)	(0.09, 0.50, 0.36, 0.18)

|	c_1	c_2	c_3
Case 1	(4.75, 4.33, 4.99, 3.02)	(2.22, 3.16, 1.12, 1.94)	(0.30, 1.78, 3.26, 0.71)
Case 2	(3.37, 0.59, 1.23, 2.73)	(2.19, 4.07, 1.71, 2.81)	(2.19, 1.62, 1.88, 1.98)
Case 3	(4.95, 0.09, 2.67, 3.13)	(0.34, 3.42, 4.43, 0.69)	(4.70, 3.92, 4.50, 1.09)
Case 4	(0.37, 0.97, 0.20, 2.79)	(2.95, 2.16, 4.73, 0.92)	(4.55, 3.75, 3.82, 2.49)

	A_1^*	A_2^*	A_3^*	$A_{\{1,2\}}^*$	$A_{\{2,3\}}^*$	$A_{\{1,2,3\}}^*$	$A^{*(1)}$	$A^{*(2)}$	$A^{*(3)}$
Case 1	-3.37	-0.37	-1.07	-2.090	-0.75	-1.75	-1.75	-1.75	-1.75
Case 2	-0.75	-1.88	-0.72	-1.35	-1.39	-1.17	-0.75	-1.39	-1.39
Case 3	-2.57	-0.77	-5.64	-1.73	-2.74	-2.67	-1.73	-1.73	-5.64
Case 4	1.46	-1.42	-2.56	-0.25	-2.01	-1.13	1.46	-1.42	-2.56

TABLE 1. Parameters and threshold levels for (3.26).

We now move onto verifying (3.27) or the optimality over $\widetilde{\mathcal{S}}_3$ for a wider class of f and g. We assume (b) $g_m = g^{(lin)}$ for some $\alpha = \alpha_m$ and $\beta = 0$ for each $1 \leq m \leq 3$, (i) $f_1 = \gamma_1 f^{(sim)}$ with $I_1 = (-\infty, 0)$, $I_2 = [0, \infty)$, $f^{(1)} = -10$ and $f^{(2)} = 10$, (ii) $f_2 = \gamma_2 f^{(lin)}$ for $b_1 = 1$ and $b_2 = 0$ with sufficiently small b_3 and (iii) $f_3 = \gamma_3 f^{(exp)}$ for $L = B = 1$. We simulate the values of $\{\alpha_m, \gamma_m, 1 \leq m \leq 3\}$ until we obtain each of Cases 1 to 4 as described above. The generated parameters and threshold levels are summarized in Table 2 and the comparison with perturbed strategies are given in Figure 5. We again observe in all cases that the value obtained by $(\tau_{A^{*(1)}}, \tau_{A^{*(2)}}, \tau_{A^{*(3)}})$ indeed dominates uniformly over x those obtained by the perturbed strategies. This verifies numerically (3.27) and we also observe the same smoothness properties as in Figure 4.

5. CONCLUDING REMARKS

In this paper, we studied a wide class of optimal stopping problems for a general spectrally negative Lévy process and extended them to multiple-stopping. Our framework is applicable to a wide range of settings particularly in real option problems where the firm withdraw from a project in stages. Our analytical results suggest that the optimal solutions can be characterized by the threshold levels that make certain monotone functions vanish and the corresponding value functions can be expressed in terms of the scale function. Our numerical experiments suggest,
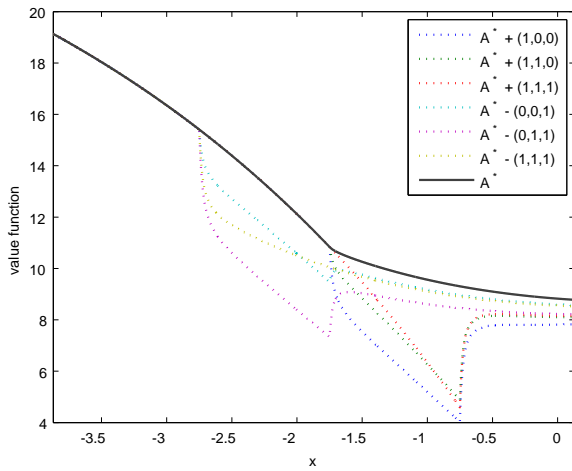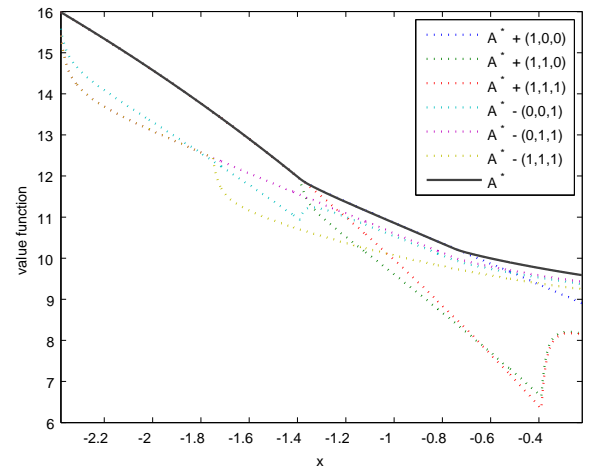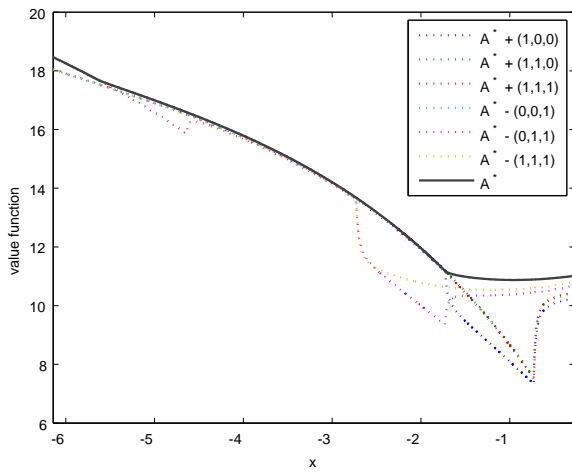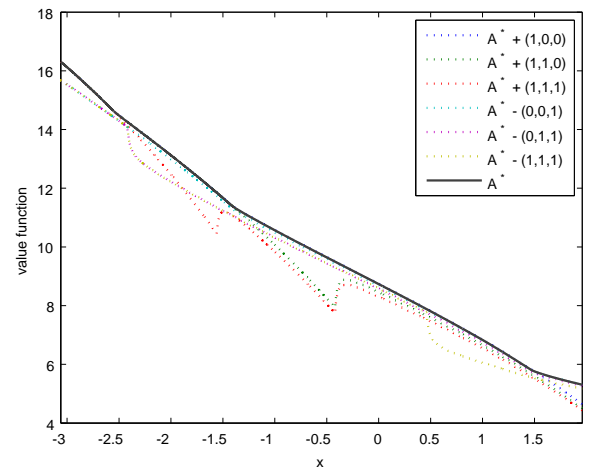
FIGURE 4. Verification of optimality (3.26) for the multiple-stage case: the value function $U^{(3)}$ (solid) in comparison with the perturbed strategies.

for the hyperexponential jump case, that these can be solved instantaneously with high precision. These tools we developed in this paper are highly valuable and can be used flexibly for analysis in real options and other fields of finance and industrial applications.

There are several directions for future research. First, our results can be pursued for a general Lévy process with both positive and negative jumps. While it makes the problem less tractable, it is expected that these can be done at least for the cases with analytical forms of Wiener-Hopf factors such as double exponential jump diffusion [20] and phase-type Lévy processes [2]. Second, by using hyperexponential fitting, one can approximate any Lévy process with completely monotone jumps (such as CGMY and variance gamma processes) by those with hyperexponential jumps as in Section 4. By calibrating with real financial and industrial data as in [3], one can conduct detailed

	α_1	α_2	α_3	γ_1	γ_2	γ_3
Case 1	0.0596	0.2687	0.9867	0.5824	0.4210	0.0921
Case 2	0.9119	0.0149	0.1567	0.7722	0.4754	0.6809
Case 3	0.1874	0.5317	0.3550	0.6003	0.0850	0.9224
Case 4	0.7862	0.7231	0.2788	0.6535	0.4897	0.9729

	A_1^*	A_2^*	A_3^*	$A_{\{1,2\}}^*$	$A_{\{2,3\}}^*$	$A_{\{1,2,3\}}^*$	$A^{*(1)}$	$A^{*(2)}$	$A^{*(3)}$
Case 1	-4.53	-4.71	-1.56	-4.61	-2.84	- 3.47	- 3.47	- 3.47	- 3.47
Case 2	-3.32	-6.69	-6.20	-3.93	-6.52	-4.15	-3.32	-6.52	-6.52
Case 3	-4.17	-2.12	-4.87	-3.47	-3.42	-3.79	-3.47	-3.47	-4.87
Case 4	-3.31	-3.58	-5.60	-3.41	-4.14	-3.70	-3.31	-3.58	-5.60

TABLE 2. Parameters and threshold levels for (3.27).

empirical analyses on optimal stopping strategies and the value functions. Finally, it is an interesting extension to "swing option type" multiple-stopping with refracting time as in [10, 11] where there have to be some intervals between any two stoppings.

APPENDIX A. PROOFS

A.1. **Proof of Lemma 2.1.** Substituting (2.20) in (2.11),

$$
\text{(A.1)} \qquad \rho_{g,A}^{(r)} = -\sum_{i=1}^{N} c_i e^{a_i A} \int_0^\infty \Pi(\mathrm{d}u) \int_0^u e^{-\Phi_r z}(e^{a_i(z-u)} - 1)\mathrm{d}z.
$$

(Case 1) First suppose $a_i \neq \Phi_r$ for all $1 \leq i \leq N$. Simple algebra gives
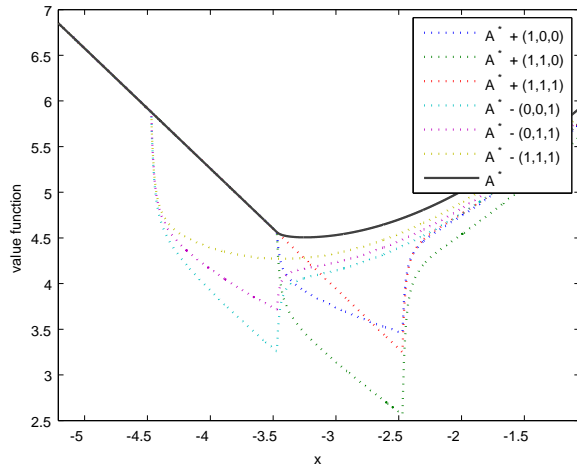
$$
\text{(A.2)} \qquad \Lambda(A) = -\frac{r}{\Phi_r} K + \sum_{i=1}^{N} c_i e^{a_i A} \frac{M_r^{(a_i)}}{\Phi_r} + \Psi_f(A)
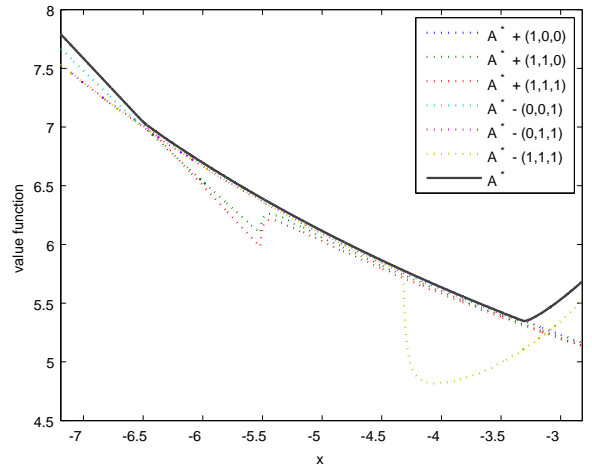$$

where

$$
M_r^{(a)} := r + \frac{a\sigma^2}{2}\Phi_r + \int_0^\infty \Pi(\mathrm{d}u)\Big[(1 - e^{-\Phi_r u}) - e^{-au}(1 - e^{-(\Phi_r - a)u})\frac{\Phi_r}{\Phi_r - a}\Big], \quad a \in \mathbb{R}\backslash\{\Phi_r\}.
$$
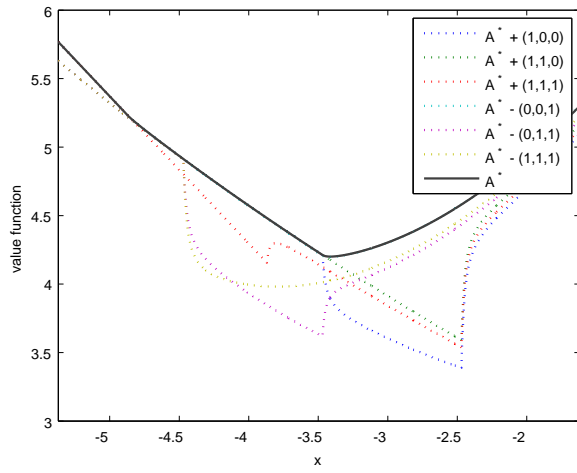
By the definition of ψ and Φ_r, we rewrite $M_r^{(a)}$ as

$$
r + \frac{a\sigma^2}{2}\Phi_r + \int_0^\infty \Pi(\mathrm{d}u)\Big[(1 - e^{-\Phi_r u} - \Phi_r u 1_{\{u \in (0,1)\}}) - e^{-au}(1 - e^{-(\Phi_r - a)u})\frac{\Phi_r}{\Phi_r - a} + \Phi_r u 1_{\{u \in (0,1)\}}\Big]
$$

$$
= \Big(c - \int_0^1 u(e^{-au} - 1)\Pi(\mathrm{d}u)\Big)\Phi_r + \frac{\sigma^2}{2}\Phi_r(\Phi_r + a)
$$

$$
- \frac{\Phi_r}{\Phi_r - a}\int_0^\infty \Pi(\mathrm{d}u)e^{-au}\Big(1 - e^{-(\Phi_r - a)u} + (\Phi_r - a)u 1_{\{u \in (0,1)\}}\Big)
$$

$$
= \frac{\Phi_r}{\Phi_r - a}\psi_a(\Phi_r - a),
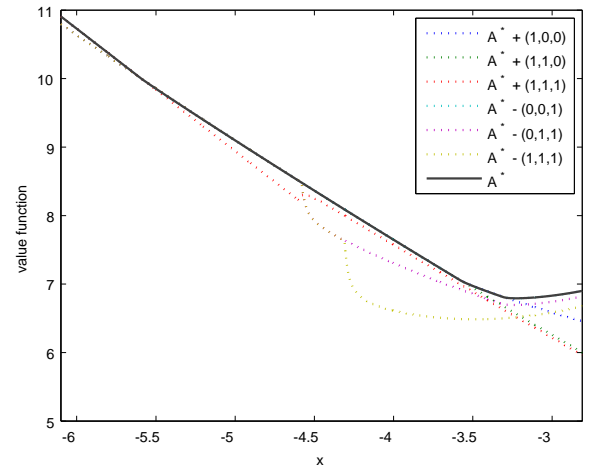$$

FIGURE 5. Verification of optimality (3.27) for the multiple-stage case: the value function $\widetilde{U}^{(3)}$ (solid) in comparison with the perturbed strategies.

where the last equality holds by (2.5). On the other hand, $\psi_a(\Phi_r - a) = \psi(\Phi_r) - \psi(a) = r - \psi(a)$; see page 213 of [21]. Hence

$$\Phi_r \varpi_r(a) = \frac{\Phi_r}{\Phi_r - a}(r - \psi(a)) = \frac{\Phi_r}{\Phi_r - a}\psi_a(\Phi_r - a),$$

which shows for the case $a_i \neq \Phi_r$ for all $1 \leq i \leq N$.

(Case 2) Suppose $a_j = \Phi_r$ for some $1 \leq j \leq N$ (with $a_i \neq a_j$ for $i \neq j$ by assumption). Take a sequence of (strictly) increasing sequence $a_j^{(m)} \uparrow a_j = \Phi_r$. Then a modification of (2.17) with a_j replaced with $a_j^{(m)}$ is by

Case 1

$$\Lambda^{(m)}(A) = -\frac{r}{\Phi_r}K + \sum_{1 \le i \le N, i \ne j} c_i e^{a_i A} \varpi_r(a_i) + c_j e^{a_j^{(m)} A} \varpi_r(a_j^{(m)}) + \Psi_f(A).$$

By the definition of ϖ_r as in (2.21), we have

$$\lim_{m \uparrow \infty} \Lambda^{(m)}(A) = -\frac{r}{\Phi_r}K + \sum_{i=1}^{N} c_i e^{a_i A} \varpi_r(a_i) + \Psi_f(A).$$

On the other hand, in view of (A.1), its integrand is monotone in a. Hence by the monotone convergence theorem and because g and g' are continuous in a, $\lim_{m \uparrow \infty} \Lambda^{(m)}(A) = \Lambda(A)$, and the proof is complete for Case 2.

A.2. **Proof of Lemma 2.2.** Because $g(x)$ is infinitely differentiable, the results are clear for $x \in (-\infty, A^*)$. Hence we show for $x \in (A^*, \infty)$. Because $W^{(r)}(y)$ is differentiable on $y > 0$ as in Remark 2.1 (1), $KZ^{(r)}(x - A^*) - \sum_{i=1}^{N} c_i e^{a_i x} Z_{a_i}^{(r-\psi(a_i))}(x - A^*)$ is twice differentiable.

Regarding $\Theta_f(x; A^*)$, as in the proof of Lemma 4.5 of [16], integration by parts, thanks to the continuity of f, gives

$$r \int_{A^*}^{x} W^{(r)}(x - y) f(y) \mathrm{d}y$$

$$= \left[f(y)(Z^{(r)}(x) - Z^{(r)}(x - y)) \right]_{y=A^*}^{y=x} - \int_{A^*}^{x} f'(y)(Z^{(r)}(x) - Z^{(r)}(x - y)) \mathrm{d}y$$

$$= f(x)(Z^{(r)}(x) - 1) - f(A^*)(Z^{(r)}(x) - Z^{(r)}(x - A^*)) - (f(x) - f(A^*))Z^{(r)}(x) + \int_{A^*}^{x} f'(y)Z^{(r)}(x - y) \mathrm{d}y$$

$$= -f(x) + f(A^*)Z^{(r)}(x - A^*) + \int_{A^*}^{x} f'(y)Z^{(r)}(x - y) \mathrm{d}y.$$

Hence

$$\Theta_f(x; A^*) = \frac{1}{r}\left[-f(x) + f(A^*)Z^{(r)}(x - A^*) + \int_{A^*}^{x} f'(y)Z^{(r)}(x - y) \mathrm{d}y \right].$$

Because $Z^{(r)}(0) = 1$, $\Theta_f(x; A^*)$ is differentiable with

$$\Theta_f'(x; A^*) = f(A^*)W^{(r)}(x - A^*) + \int_{A^*}^{x} f'(y)W^{(r)}(x - y) \mathrm{d}y.$$

When X is of unbounded variation, because $W^{(r)}(0) = 0$ as in Remark 2.1 (2), $\Theta_f(x; A^*)$ is twice-differentiable with

$$\Theta_f''(x; A^*) = f(A^*)W^{(r)'}(x - A^*) + \int_{A^*}^{x} f'(y)W^{(r)'}(x - y) \mathrm{d}y.$$

A.3. **Proof of Proposition 2.1.** Suppose $-\infty < A^* < \infty$. By directly using the results of [17] (Lemma 3.7 and Proposition 3.4), we obtain

$$(\mathcal{L} - r)u_{A^*}(x) + f(x) = 0, \quad x \in (A^*, \infty),$$

$$u_{A^*}(x) \ge g(x), \quad x \in \mathbb{R},$$

where $\mathcal{L}$ is the infinitesimal generator of X applied to a sufficiently smooth function h, i.e.,

$$\mathcal{L}h(x) = ch'(x) + \frac{1}{2}\sigma^2 h''(x) + \int_0^\infty [h(x-z) - h(x) + h'(x)z1_{\{0<z<1\}}]\Pi(\mathrm{d}z).$$

We shall show $(\mathcal{L}-r)u_{A^*}(x) + f(x) < 0$ on $x \in (-\infty, A^*)$ and verify the optimality. By the definition of ψ,

$$\mathcal{L}e^{ax} = e^{ax}\left[ca + \frac{1}{2}\sigma^2 a^2 + \int_0^\infty \left[e^{-az} - 1 + az1_{\{0<z<1\}}\right]\Pi(\mathrm{d}z)\right] = e^{ax}\psi(a)$$

for any $a > 0$ and hence we have

(A.3) $$(\mathcal{L}-r)g(x) + f(x) = -rK + \sum_{i=1}^N c_i e^{a_i x}(r - \psi(a_i)) + f(x).$$

By how A^* is chosen,

(A.4) $$0 = -rK + \sum_{i=1}^N c_i e^{a_i A^*}\Phi_r \varpi_r(a_i) + \Phi_r \Psi_f(A^*).$$

Because f is increasing and $x < A^*$

(A.5) $$\Phi_r \Psi_f(A^*) \geq \Phi_r \int_0^\infty e^{-\Phi_r y}f(x)\mathrm{d}y = f(x).$$

It is also easy to see that for any $1 \leq i \leq N$

(A.6) $$e^{a_i A^*}\Phi_r \varpi_r(a_i) \geq e^{a_i x}(r - \psi(a_i)).$$

Indeed, for the case $r - \psi(a_i) > 0$, we must have $\Phi_r - a_i > 0$ and hence (A.6) holds by $A^* > x$; for the case $r - \psi(a_i) < 0$, the left-hand side is positive while the right-hand side is negative in (A.6); for the case $r - \psi(a_i) = 0$, the left-hand side is positive because $\psi'(\Phi_r)$ is, while the right-hand side is zero. Hence, by (A.4)-(A.6), $(\mathcal{L}-r)u_{A^*}(x) + f(x) < 0$ holds.

This result also holds for the case $A^* = \infty$. In this case, $0 > -rK + \sum_{i=1}^N c_i e^{a_i \widehat{A}}\Phi_r \varpi_r(a_i) + \Phi_r \Psi_f(\widehat{A})$, for any $\widehat{A} \in \mathbb{R}$. Therefore $(\mathcal{L}-r)g(x) + f(x) < 0$ holds by the same reasoning as in (1) by simply replacing A^* with $\widehat{A}$.

Finally, we verify the optimality of $u_{A^*}(x)$. Thanks to Lemma 2.2 and the continuous/smooth fit condition as in Remark 2.4, a version of Meyer-Ito's formula as in Theorem IV.71 of [28] (see also Theorem 2.1 of [27]) implies $\{e^{-rt}u_{A^*}(X_t) - \int_0^t e^{-rs}(\mathcal{L}-r)u_{A^*}(X_s)1_{\{X_s \neq A^*\}}\mathrm{d}s, \ t \geq 0\}$ is a local martingale. Namely, there exists some localizing sequence $\{\sigma_m, m \geq 1\}$ such that $\{e^{-r(t\wedge\sigma_m)}u_{A^*}(X_{t\wedge\sigma_m}) - \int_0^{t\wedge\sigma_m} e^{-rs}(\mathcal{L}-r)u_{A^*}(X_s)1_{\{X_s \neq A^*\}}\mathrm{d}s, \ t \geq 0\}$ is a martingale for any $m \geq 1$. By the optional sampling theorem, for any stopping time $\tau \in \mathcal{S}$, we have for any $L > 0$

$$\mathbb{E}^x\left[e^{-r(\tau\wedge\sigma_m\wedge L)}u_{A^*}(X_{\tau\wedge\sigma_m\wedge L}) - \int_0^{\tau\wedge\sigma_m\wedge L} e^{-rs}(\mathcal{L}-r)u_{A^*}(X_s)1_{\{X_s \neq A^*\}}\mathrm{d}s\right] = u_{A^*}(x).$$

Because $(\mathcal{L}-r)u_{A^*} + f \leq 0$,

$$\mathbb{E}^x\left[e^{-r(\tau\wedge\sigma_m\wedge L)}u_{A^*}(X_{\tau\wedge\sigma_m\wedge L}) + \int_0^{\tau\wedge\sigma_m\wedge L} e^{-rs}f(X_s)\mathrm{d}s\right] \leq u_{A^*}(x).$$

Moreover, because $u_{A^*}(x) \geq g(x)$ and $u_{A^*}(x) \geq \lim_{A\downarrow-\infty} u_A(x) \geq \int_0^\infty e^{-rt}f(-\infty)\mathrm{d}t =: B$ uniformly by (2.19), we have

$$\mathbb{E}^x \left[e^{-r(\tau \wedge \sigma_m \wedge L)}(g(X_{\tau \wedge \sigma_m \wedge L}) \vee B) + \int_0^{\tau \wedge \sigma_m \wedge L} e^{-rs}f(X_s)\mathrm{d}s \right] \leq u_{A^*}(x).$$

Because $g \vee B \in [B, K \vee B]$, by the dominated convergence theorem

$$\lim_{m\uparrow\infty} \lim_{L\uparrow\infty} \mathbb{E}^x \left[e^{-r(\tau \wedge \sigma_m \wedge L)}(g(X_{t\wedge\sigma_m\wedge L}) \vee B) \right] = \mathbb{E}^x \left[e^{-r\tau}(g(X_\tau) \vee B) \right] \geq \mathbb{E}^x \left[e^{-r\tau}g(X_\tau) \right].$$

On the other hand, because $f(-\infty) > -\infty$ and by the monotone convergence theorem,

$$\lim_{m\uparrow\infty} \lim_{L\uparrow\infty} \mathbb{E}^x \left[\int_0^{\tau\wedge\sigma_m\wedge L} e^{-rs}f(X_s)\mathrm{d}s \right]$$

$$= \lim_{m\uparrow\infty} \lim_{L\uparrow\infty} \mathbb{E}^x \left[\int_0^{\tau\wedge\sigma_m\wedge L} e^{-rs}f_+(X_s)\mathrm{d}s \right] - \lim_{m\uparrow\infty} \lim_{L\uparrow\infty} \mathbb{E}^x \left[\int_0^{\tau\wedge\sigma_m\wedge L} e^{-rs}f_-(X_s)\mathrm{d}s \right] = \mathbb{E}^x \left[\int_0^\tau e^{-rs}f(X_s)\mathrm{d}s \right].$$

This means $\mathbb{E}^x \left[e^{-r\tau}g(X_\tau) + \int_0^\tau e^{-rs}f(X_s)\mathrm{d}s \right] \leq u_{A^*}(x)$, for any arbitrary $\tau \in \mathcal{S}$, as desired.

It is now left to show for the case $A^* = -\infty$. Because $\partial u_A(x)/\partial A < 0$ for any $A \in \mathbb{R}$ as in (2.19), there exists

$$u_{-\infty}(x) := \lim_{A\downarrow-\infty} u_A(x) = \lim_{A\downarrow-\infty} \mathbb{E}^x \left[\int_0^{\tau_A} e^{-rt}f(X_t)\mathrm{d}t \right].$$

By Assumption 2.2 (1) and by the monotone convergence theorem, we can write

$$u_{-\infty}(x) = \lim_{A\downarrow-\infty} \left(\mathbb{E}^x \left[\int_0^{\tau_A} e^{-rt}f_+(X_t)\mathrm{d}t \right] - \mathbb{E}^x \left[\int_0^{\tau_A} e^{-rt}f_-(X_t)\mathrm{d}t \right] \right)$$

$$= \mathbb{E}^x \left[\int_0^\infty e^{-rt}f_+(X_t)\mathrm{d}t \right] - \mathbb{E}^x \left[\int_0^\infty e^{-rt}f_-(X_t)\mathrm{d}t \right] = \mathbb{E}^x \left[\int_0^\infty e^{-rt}f(X_t)\mathrm{d}t \right].$$

Now by Corollary 8.9 of [21], $u_{-\infty}$ attains the proposed form. Because $\partial u_A(x)/\partial A < 0$ for any $A \in \mathbb{R}$, clearly $u_{-\infty}(x) > g(x)$ for any $x \in \mathbb{R}$. Moreover, because $u_{-\infty}$ is attains by $\tau^* = \infty$, we have the claim.

A.4. **Proof of Proposition 2.2.** (1,2) We first suppose $x < A^*$. Then by definition $u_{A^*}(x) = g(x)$. Because $u_A(x) = g(x)$ for any $A \geq x$, it is sufficient to show $u_A(x) \leq g(x)$ for $A < x$. This is indeed so because by (2.16), (2.19) and $\Lambda(x) < 0$ due to $x < A^*$,

$$u_A(x) \leq u_x(x+) = g(x) + W^{(r)}(0)\Lambda(x) \leq g(x) = u_{A^*}(x), \quad A < x < A^*.$$

This proves (2). For (1), suppose $x \geq A^*$. Then by (2.19) $u_{A^*}(x) \geq u_A(x)$ for any $A \leq x$. For $A \geq x$, $u_A(x) = g(x)$ and by (2.16), (2.19) and $\Lambda(x) > 0$ due to $x > A^*$,

$$u_{A^*}(x) \geq u_x(x+) = g(x) + W^{(r)}(0)\Lambda(x) \geq g(x) = u_A(x), \quad A \geq x > A^*.$$

Therefore $u_{A^*}(x) \geq u_A(x)$ uniformly in $x \in \mathbb{R}$, as desired.

The corresponding value function (for (1)) can be expressed as the sum of (2.14):

$$\widetilde{u}(x) = g(A^*)Z^{(r)}(x - A^*) + W^{(r)}(x - A^*)\left(-\frac{r}{\Phi_r}g(A^*) + \rho_{g,A^*}^{(r)} + \Psi_f(A^*) \right) - \varphi_{g,A^*}^{(r)}(x) - \Theta_f(x; A^*).$$

From the definition of A^* that makes (2.17) vanish,

$$\widetilde{u}(x) = g(A^*)Z^{(r)}(x - A^*) + W^{(r)}(x - A^*)\frac{\sigma^2}{2}g'(A^*) - \varphi_{g,A^*}^{(r)}(x) - \Theta_f(x; A^*),$$

as desired.

(3) the proof is the same as that of Proposition 2.1 (3).

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