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TOWARD A GENERALIZATION OF THE LELAND-TOFT OPTIMAL CAPITAL STRUCTURE MODEL*

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ABSTRACT. The optimal capital structure model with endogenous bankruptcy was first studied by Leland [14] and Leland and Toft [15], and was later extended to the spectrally negative Lévy model by Hilberink and Rogers [8] and Kyprianou and Surya [12]. This paper generalizes the problem by allowing the values of bankruptcy costs, coupon rates and tax benefits dependent on the firm's asset value. By using the fluctuation identities for the spectrally negative Lévy process, we obtain a candidate bankruptcy level as well as a sufficient condition for optimality. The optimality holds in particular when, monotonically in the asset value, the coupon rate is decreasing, the value of tax benefits is increasing, the loss amount at bankruptcy is increasing, and its proportion relative to the asset value is decreasing. The solution admits a semi-explicit form, and this allows for instant computation of the optimal bankruptcy levels, equity/debt values and optimal leverage ratios.

Keywords: Credit risk, optimal capital structure, spectrally negative Lévy processes, scale functions

JEL Classification: D92, G32, G33

Mathematics Subject Classification (2010): 60G40, 60G51, 91G40

1. INTRODUCTION

We revisit the Leland-Toft optimal capital structure model [14, 15] of determining the optimal endogenous bankruptcy levels. A firm is partly financed by debt of equal seniority that is continuously retired and reissued so that its maturity profile is kept constant through time. It distributes a continuous stream of coupon payments to bondholders, on which the firm receives tax benefits when its asset value is sufficiently high. The objective for the shareholders is to determine the optimal bankruptcy level so as to maximize the firm's equity value subject to the liability constraint that it never goes below zero.

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This problem was first studied by Leland [14] and Leland and Toft [15] where they assumed geometric Brownian motion for the firm's asset value, and was later extended to a Lévy model by Chen and Kou [5], Hilberink and Rogers [8] and Kyprianou and Surya [12]. By introducing jumps, it allows the value of bankruptcy costs to be stochastic, and more importantly resolves the contradictory conclusion under the continuous diffusion model that the credit spreads go to zero as the maturity decreases to zero. The problem reduces to a non-standard optimal stopping problem, and its solution can be obtained via the continuous/smooth fit principle. It was solved, in particular, for the double exponential jump diffusion process [5] and for a general spectrally negative Lévy process [8, 12].

In this paper, we generalize the problem and pursue optimal solutions. Despite the fascinating contributions of the aforementioned papers, there are several assumptions on the bankruptcy costs, coupon rates and tax benefits, which are rather artificially imposed to derive explicit/analytical solutions. This paper is aimed to relax these assumptions.

Regarding the bankruptcy costs, it is commonly assumed that the amount of bankruptcy costs is a predetermined constant fraction of the firm's asset value at bankruptcy. As is clear from the fact that the recovery rate is an unpredictable value known only after bankruptcy, this is unfortunately an unrealistic assumption. It is, in particular, overestimating the costs when the asset value is high; as the asset value increases, the efficiency of reorganization improves and hence the fraction of loss relative to the asset value tends to decrease. On the other hand, as addressed in the footnotes of [14], the value of bankruptcy costs is sometimes modeled as a predetermined fixed value (as opposed to a fixed fraction of the asset value). These simplifications take great roles in obtaining explicit solutions but do not fully reflect the reality.

As for the coupon rate, it is usually assumed to be constant through time. However, it is natural to consider its dependency on the firm's financial conditions that fluctuate over time. When the firm is not well performed, higher coupon payments must be paid to the bondholder in order to compensate for the higher risk of default. Naturally, coupon payments tend to be less for financially healthy firms than those with higher risk of insolvency. Regarding the tax rebate, its value is typically assumed to be some step function; it is some constant value when the firm's asset value is above a certain cut-off level and it is zero otherwise. However, this is what Hilberink and Rogers [8] call an idealization in their paper, and in addition the taxation system varies across countries and across industries. There is clearly a need for more realistic and flexible models.

This paper resolves these inflexibilities of the existing models by expressing the values of bankruptcy costs, coupon rates and tax benefits as functions of the firm's asset value. This generalization encompasses the existing models and adds flexibility in describing a more realistic capital structure.

We focus on the spectrally negative Lévy model considered by [8, 12] where the firm's asset value is driven by a general Lévy process with only negative jumps. Following the procedure by [12], we take advantage of the fluctuation identities expressed via the scale function. In particular, we generalize [12] using the formulas given in Egami and Yamazaki [7]. We obtain a sufficient condition for optimality and show the optimality for example when, monotonically in the asset value, the coupon rate is decreasing, the value of tax benefits is increasing, the loss amount at bankruptcy is increasing, and its proportion relative to the asset value is decreasing.

The solutions to our generalized model admit semi-explicit expressions written in terms of the scale function, which has analytical forms in certain cases [7, 9, 11, 12] and can be approximated generally using, e.g., [6, 16]. In order to illustrate the implementation side, we give an example based on a mixture of Brownian motion and a compound Poisson process with i.i.d. exponential jumps. We compute the optimal bankruptcy levels and the corresponding debt/equity/firm values as well as the optimal leverage ratios as solutions to the *two-stage problem* considered in [5, 14, 15].

The rest of the paper is organized as follows. In Section 2, we review the existing Lévy model and introduce our generalized model. In Section 3, we focus on the spectrally negative Lévy model and obtain a sufficient condition for optimality. Section 4 shows examples that satisfy the sufficient condition. We give numerical results in Section 5 and concluding remarks in Section 6.

2. Problem Formulation

In this section, we first review the existing optimal capital structure model and then generalize it. In particular, we adopt the formulation and notations by [8, 12] to a maximum extent. The formulation addressed here holds for any Lévy model; in the next section, we focus on the spectrally negative Lévy process and derive optimal solutions.

2.1. **The optimal capital structure model by [8, 12].** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space hosting a Lévy process $X = \{X_t;\, t \geq 0\}$. The value of the *firm's asset* is assumed to evolve according to an *exponential Lévy process* $V_t := e^{X_t}$, $t \geq 0$. Let $r > 0$ be the positive risk-free interest rate and $0 \leq \delta < r$ the total payout rate to the firm's investors. We assume that the market is complete and this requires $\{e^{-(r-\delta)t} V_t; t \geq 0\}$ to be a $\mathbb{P}$-martingale. We denote by $\mathbb{P}_x$ the probability law and $\mathbb{E}_x$ the expectation under which $X_0 = x$ (or equivalently $V_0 = e^x$).

The firm is partly financed by debt with a constant debt profile; it issues new debt at a constant rate p with maturity profile $\varphi(s) := m e^{-ms}$ for some given constants $p, m > 0$. Namely, in the time interval $(t, t + \mathrm{d}t)$, it issues debt with face value $p\varphi(s)\mathrm{d}t\mathrm{d}s$ that matures in the time interval $(t + s, t + s + \mathrm{d}s)$. By this assumption, at time 0, the face value of debt that matures in $(s, s + \mathrm{d}s)$ becomes

$$\left[\int_{-\infty}^{0} p\varphi(s-u)\mathrm{d}u\right]\mathrm{d}s = pe^{-ms}\mathrm{d}s, \tag{2.1}$$

and the face value of all debt is a constant value,

$$P := \int_0^\infty pe^{-ms}\mathrm{d}s = \frac{p}{m}.$$

For more details, see [8, 12].

Suppose the bankruptcy is triggered at the first time X goes below a given level $B \in \mathbb{R}$, or

$$\tau_B^- := \inf \{t \geq 0 : X_t < B\}, \quad B \in \mathbb{R}. \tag{2.2}$$

The debt pays a constant coupon flow at a fixed rate $\hat{\rho} > 0$ and a constant fraction $0 \leq \hat{\eta} \leq 1$ of the asset value is lost at the bankruptcy time τ_B^-; the value of the debt with a unit face value and maturity $t > 0$ becomes

$$d(x; B, t) := \mathbb{E}_x \left[\int_0^{t \wedge \tau_B^-} e^{-rs} \hat{\rho} \mathrm{d}s \right] + \mathbb{E}_x \left[e^{-rt} 1_{\{t < \tau_B^-\}} \right] + \frac{1}{P} \mathbb{E}_x \left[e^{-r\tau_B^- + X_{\tau_B^-}} (1 - \hat{\eta}) 1_{\{\tau_B^- < t\}} \right]. \tag{2.3}$$

Here, the first term is the total value of the coupon payments accumulated until maturity or bankruptcy whichever comes first. The second term is the value of the principle payment. The last term corresponds to the $1/P$ fraction of the asset value that is distributed, in the event of bankruptcy, to the bondholder of a unit face value. The *total value of debt* becomes, by (2.1) and Fubini's theorem,

$$\begin{aligned} \mathcal{D}(x; B) &:= \int_0^\infty p e^{-mt} d(x; B, t) \mathrm{d}t \\ &= \mathbb{E}_x \left[\int_0^{\tau_B^-} e^{-(r+m)t} (P\hat{\rho} + p) \, \mathrm{d}t \right] + \mathbb{E}_x \left[e^{-(r+m)\tau_B^- + X_{\tau_B^-}} (1 - \hat{\eta}) 1_{\{\tau_B^- < \infty\}} \right]. \end{aligned}$$

Regarding the *(market) value of the firm*, it is assumed that there is a corporate tax rate $\hat{\gamma} > 0$ and its (full) rebate on coupon payments is gained if and only if $V_t \geq V_T$ (or $X_t \geq \log V_T$) for some given cutoff level $V_T > 0$. Based on the Modigliani-Miller theorem (see e.g. [3]), the firm value becomes

$$\mathcal{V}(x; B) := e^x + \mathbb{E}_x \left[\int_0^{\tau_B^-} e^{-rt} 1_{\{X_t \geq \log V_T\}} P \hat{\gamma} \hat{\rho} \mathrm{d}t \right] - \hat{\eta} \mathbb{E}_x \left[e^{-r\tau_B^- + X_{\tau_B^-}} 1_{\{\tau_B^- < \infty\}} \right],$$

where each term corresponds to the current asset value, the total value of tax benefits and the value of loss at bankruptcy, respectively.

The problem is to pursue an *optimal bankruptcy level* $B \in \mathbb{R}$ that maximizes the *equity value*,

$$\mathcal{E}(x; B) := \mathcal{V}(x; B) - \mathcal{D}(x; B), \quad x > B, \tag{2.4}$$

subject to the *limit liability constraint*,

$$\mathcal{E}(x; B) \geq 0, \quad x \geq B, \tag{2.5}$$

if such a level exists. This Lévy model was first solved by Hilberink and Rogers [8] for a special class of Lévy processes taking the form of an independent sum of a linear Brownian motion and a compound Poisson process with negative jumps (cf. (3.21) on page 245 of [8]). Kyprianou and Surya [12] later showed for a general spectrally negative process that the optimal bankruptcy level exists and is explicitly determined by applying continuous and smooth fit when X is of bounded and unbounded variation, respectively.

2.2. **Our generalization.** We now generalize the model described above by allowing the fraction of loss $\hat{\eta}$, coupon rate $\hat{\rho}$, and tax rebate rate $\hat{\gamma}$ dependent on X.

First, we generalize the debt value (2.3) to

$$d(x; B, t) := \mathbb{E}_x \left[\int_0^{t \wedge \tau_B^-} e^{-rs} \rho(X_s) \mathrm{d}s \right] + \mathbb{E}_x \left[e^{-rt} 1_{\{t < \tau_B^-\}} \right] + \frac{1}{P} \mathbb{E}_x \left[e^{-r\tau_B^- + X_{\tau_B^-}} \left(1 - \overline{\eta}(X_{\tau_B^-})\right) 1_{\{\tau_B^- < t\}} \right]$$

where $\rho(\cdot) \geq 0$ and $\overline{\eta}(\cdot) \geq 0$ are the coupon rate and the rate of loss at bankruptcy, respectively. The total debt value becomes

$$\mathcal{D}(x; B) := \mathbb{E}_x \left[\int_0^{\tau_B^-} e^{-(r+m)t} \left(P\rho(X_t) + p \right) \mathrm{d}t \right] + \mathbb{E}_x \left[e^{-(r+m)\tau_B^- + X_{\tau_B^-}} \left(1 - \overline{\eta}(X_{\tau_B^-})\right) 1_{\{\tau_B^- < \infty\}} \right].$$

By setting

$$f_1(y) := P\rho(y) + p \quad \text{and} \quad \eta(y) := e^y \overline{\eta}(y), \quad y \in \mathbb{R},$$

we can write

$$\mathcal{D}(x; B) = \mathbb{E}_x \left[\int_0^{\tau_B^-} e^{-(r+m)t} f_1(X_t) \mathrm{d}t \right] + \mathbb{E}_x \left[e^{-(r+m)\tau_B^- + X_{\tau_B^-}} 1_{\{\tau_B^- < \infty\}} \right] - \mathbb{E}_x \left[e^{-(r+m)\tau_B^-} \eta(X_{\tau_B^-}) 1_{\{\tau_B^- < \infty\}} \right]. \tag{2.6}$$

Here notice that $\eta(\cdot)$ denotes the total loss amount whereas $\overline{\eta}(\cdot)$ is the rate of loss relative to the asset value. We allow $\overline{\eta}$ to be larger than 1, which lets one to model, for example, the case η is constant; see Section 4.3 below.

We next generalize the tax rebates; the firm value with default level B is

$$\mathcal{V}(x; B) := e^x + \mathbb{E}_x \left[\int_0^{\tau_B^-} e^{-rt} f_2(X_t) \mathrm{d}t \right] - \mathbb{E}_x \left[e^{-r\tau_B^-} \eta(X_{\tau_B^-}) 1_{\{\tau_B^- < \infty\}} \right], \tag{2.7}$$

where $f_2(\cdot) \geq 0$ is the rate of tax rebates. As we discuss in Remark 3.1 below, under Assumptions 3.2-3.3, each expectation in (2.6)-(2.7) is finite for all $x > B$ and hence $\mathcal{E}(x; B) = \mathcal{V}(x; B) - \mathcal{D}(x; B)$ is well-defined.

3. Solutions for the Spectrally negative Lévy models

In this section, we study the problem (2.4)-(2.5) with the debt and firm values generalized to (2.6)-(2.7) focusing on the case X is a spectrally negative Lévy process, or a Lévy process with only negative jumps. We assume that X is uniquely defined by the *Laplace exponent*,

$$\kappa(s) := \log \mathbb{E}_0 \left[e^{sX_1} \right] = cs + \frac{1}{2}\sigma^2 s^2 + \int_{(0,\infty)} (e^{-sx} - 1 + sx 1_{\{0 < x < 1\}}) \, \Pi(\mathrm{d}x), \quad s \in \mathbb{R}, \tag{3.1}$$

where $c \in \mathbb{R}$, $\sigma \geq 0$, and Π is a measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty.$$

We ignore the case X is a negative subordinator (monotonically decreasing a.s.).

The process X has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(0,\infty)}(1 \wedge x)\Pi(\mathrm{d}x) < \infty$. For more details about the spectrally negative Lévy process, we refer the reader to, e.g., [2, 10].

3.1. **Scale functions.** For our derivation of optimal solutions, we first rewrite (2.6)-(2.7) using the scale function. For a given spectrally negative Lévy process with Laplace exponent κ, there exists a continuous and increasing function

$$W^{(q)} : \mathbb{R} \mapsto \mathbb{R}_+; \quad q \geq 0,$$

such that $W^{(q)}(x) = 0$ for all $x < 0$ and

$$\int_0^\infty e^{-sx} W^{(q)}(x)\mathrm{d}x = \frac{1}{\kappa(s) - q}, \qquad s > \Phi(q)$$

where

$$\Phi(q) := \sup\{s > 0 : \kappa(s) = q\}, \quad q \geq 0.$$

It is known that κ is zero at the origin and strictly convex on $[0, \infty)$. Therefore $\Phi(q)$ is strictly increasing in q.

If τ_a^+ is the first time the process goes above $a > x > 0$ and τ_0^- is the first time it goes below zero as a special case of (2.2), then we have

$$\mathbb{E}_x\left[e^{-q\tau_a^+} 1_{\{\tau_a^+ < \tau_0^-, \tau_a^+ < \infty\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)},$$

$$\mathbb{E}_x\left[e^{-q\tau_0^-} 1_{\{\tau_a^+ > \tau_0^-, \tau_0^- < \infty\}}\right] = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)},$$

where

$$Z^{(q)}(x) := 1 + q\int_0^x W^{(q)}(y)\mathrm{d}y, \quad x \in \mathbb{R}.$$

Because $W^{(q)}(x) = 0$ for all $x < 0$, we have that $Z^{(q)}(x) = 1$ on $(-\infty, 0]$.

In particular, $W^{(q)}$ is continuously differentiable on $(0, \infty)$ if Π does not have atoms (see [13]), and it is twice-differentiable on $(0, \infty)$ if $\sigma > 0$ (see [4]). For the rest of this paper, we assume the former.

Assumption 3.1. *We assume that* Π *does not have atoms.*

Fix $q > 0$. The scale function increases exponentially;

$$W^{(q)}(x) \sim \frac{e^{\Phi(q)x}}{\kappa'(\Phi(q))} \quad \text{as} \ \ x \to \infty. \tag{3.2}$$

There exists a (scaled) version of the scale function $W_{\Phi(q)} = \{W_{\Phi(q)}(x); x \in \mathbb{R}\}$ that satisfies

$$W_{\Phi(q)}(x) = e^{-\Phi(q)x} W^{(q)}(x), \quad x \in \mathbb{R}$$

and

$$\int_0^\infty e^{-sx} W_{\Phi(q)}(x) \mathrm{d}x = \frac{1}{\kappa(s+\Phi(q)) - q}, \quad s > 0.$$

Moreover $W_{\Phi(q)}(x)$ is increasing, and as is clear from (3.2),

$$W_{\Phi(q)}(x) \nearrow \frac{1}{\kappa'(\Phi(q))} \quad \text{as } x \to \infty.$$

As in Lemmas 4.3-4.4 of [12], for all $q > 0$,

$$W^{(q)}(0) = \left\{ \begin{array}{ll} 0, & \text{unbounded variation} \\ \frac{1}{\mu}, & \text{bounded variation} \end{array} \right\},$$
$$W^{(q)\prime}(0+) = \left\{ \begin{array}{ll} \frac{2}{\sigma^2}, & \sigma > 0 \\ \infty, & \sigma = 0 \text{ and } \Pi(0,\infty) = \infty \\ \frac{q+\Pi(0,\infty)}{\mu^2}, & \text{compound Poisson} \end{array} \right\}, \tag{3.3}$$

where $\mu := c + \int_{(0,1)} x \Pi(\mathrm{d}x)$ that is finite for the case X is of bounded variation.

3.2. **In terms of the scale function.** We now rewrite (2.6)-(2.7) using the scale function. Toward this end, we introduce the following shorthand notations:

$$\Lambda^{(q)}(x;B) := \mathbb{E}_x\left[e^{-q\tau_B^-}\eta\big(X_{\tau_B^-}\big)1_{\{\tau_B^-<\infty\}}\right] \quad \text{and} \quad \mathcal{M}_i^{(q)}(x;B) := \mathbb{E}_x\left[\int_0^{\tau_B^-} e^{-qt} f_i(X_t)\mathrm{d}t\right], \quad i = 1,2,$$

for any $q > 0$ and $x > B$. These functions admit semi-explicit expressions in terms of the scale function. By Lemmas 2.1-2.3 of [7], for all $x > B$ and $q > 0$, we can write

$$\begin{aligned} \Lambda^{(q)}(x;B) &= \eta(B)\left[Z^{(q)}(x-B) - \frac{q}{\Phi(q)}W^{(q)}(x-B)\right] - W^{(q)}(x-B)H^{(q)}(B) \\ &\quad + \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u\wedge(x-B)} W^{(q)}(x-z-B)[\eta(B) - \eta(z+B-u)]\mathrm{d}z, \\ \mathcal{M}_i^{(q)}(x;B) &= W^{(q)}(x-B)G_i^{(q)}(B) - \int_B^x W^{(q)}(x-y)f_i(y)\mathrm{d}y, \quad i=1,2, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} H^{(q)}(B) &:= \int_0^\infty \Pi(\mathrm{d}u) \int_0^u e^{-\Phi(q)z}[\eta(B) - \eta(B-u+z)]\mathrm{d}z, \\ G_i^{(q)}(B) &:= \int_0^\infty e^{-\Phi(q)y} f_i(y+B)\mathrm{d}y, \quad i = 1,2. \end{aligned}$$

On the other hand, by Lemma 4.7 of [12], we have $\mathbb{E}_y\left[e^{-q\tau_0^- + X_{\tau_0^-}} 1_{\{\tau_0^- < \infty\}}\right] = e^y - \Gamma^{(q)}(y)$ for any $y > 0$ where

$$\Gamma^{(q)}(y) := \frac{\kappa(1) - q}{1 - \Phi(q)} W^{(q)}(y) + (\kappa(1) - q) e^y \int_0^y e^{-z} W^{(q)}(z) \mathrm{d}z, \quad q > 0 \text{ and } y > 0. \tag{3.5}$$

Hence, for any $x > B$,

$$\mathbb{E}_x\left[e^{-(r+m)\tau_B^- + X_{\tau_B^-}} 1_{\{\tau_B^- < \infty\}}\right] = e^B \mathbb{E}_{x-B}\left[e^{-(r+m)\tau_0^- + X_{\tau_0^-}} 1_{\{\tau_0^- < \infty\}}\right] = e^x - e^B \Gamma^{(r+m)}(x - B).$$

Putting altogether, for all $x > B$, we simplify (2.6)-(2.7) to

$$\begin{aligned} \mathcal{D}(x; B) &= e^x - e^B \Gamma^{(r+m)}(x - B) + \mathcal{M}_1^{(r+m)}(x; B) - \Lambda^{(r+m)}(x; B), \\ \mathcal{V}(x; B) &= e^x + \mathcal{M}_2^{(r)}(x; B) - \Lambda^{(r)}(x; B), \end{aligned}$$

and we obtain the equity value

$$\mathcal{E}(x; B) = e^B \Gamma^{(r+m)}(x - B) + (\mathcal{M}_2^{(r)}(x; B) - \Lambda^{(r)}(x; B)) - (\mathcal{M}_1^{(r+m)}(x; B) - \Lambda^{(r+m)}(x; B)). \tag{3.6}$$

In view of (3.4), the following assumption guarantees the finiteness of $\mathcal{M}_1^{(r+m)}(x; B)$ and $\mathcal{M}_2^{(r)}(x; B)$ for all $x > B$.

Assumption 3.2. *We assume that*

$$\int_0^\infty e^{-\Phi(r+m)y} f_1(y) \mathrm{d}y < \infty \quad \textit{and} \quad \int_0^\infty e^{-\Phi(r)y} f_2(y) \mathrm{d}y < \infty.$$

Regarding η, we assume the following for the rest of this section.

Assumption 3.3. *We assume that η is $C^2(\mathbb{R})$ and is bounded on $(-\infty, B]$ for any fixed $B \in \mathbb{R}$.*

Here the C^2 assumption is imposed for simplicity of the arguments; this can be relaxed as discussed in Remark 3.6 below.

Remark 3.1. *By Assumptions 3.2-3.3, the equity value $\mathcal{E}(x; B)$ is well-defined for any $x > B$.*

Remark 3.2. *Because η is continuous, $\Lambda^{(q)}(x; B)$ and $\mathcal{M}_i^{(q)}(x; B)$ are continuous in B on $(-\infty, x]$ for any fixed $x \in \mathbb{R}$.*

3.3. **Derivative with respect to B.** To derive the candidate bankruptcy level, we use the results in Egami and Yamazaki [7] and obtain the derivative of $\mathcal{E}(x; B)$ with respect to B. Define

$$\Theta^{(q)}(x) := W^{(q)\prime}(x) - \Phi(q) W^{(q)}(x) = e^{\Phi(q)x} W'_{\Phi(q)}(x), \quad x > 0 \text{ and } q > 0, \tag{3.7}$$

which is always positive. Also as in [12], $\Theta^{(q)}(x)$ is monotonically decreasing in q for every fixed $x > 0$; see [12] for an interpretation of $\Theta^{(q)}$ as the resolvent measure of the ascending ladder height process of X.

The derivatives of $\Lambda^{(q)}(x;B)$ and $\mathcal{M}_i^{(q)}(x;B)$ with respect to B require technical details. However, as shown by [7], each term can be expressed as a product of $\Theta^{(q)}(x-B)$ and some function of B (that is independent of x). For the proof of the following lemma, see the proof of Proposition 3.1 of [7].

Lemma 3.1. *For every $x > B$ and $q > 0$, we have*

$$\frac{\partial}{\partial B}\Lambda^{(q)}(x;B) = \Theta^{(q)}(x-B)\left[\frac{q}{\Phi(q)}\eta(B) + H^{(q)}(B) + \frac{\sigma^2}{2}\eta'(B)\right],$$
$$\frac{\partial}{\partial B}\mathcal{M}_i^{(q)}(x;B) = -\Theta^{(q)}(x-B)G_i^{(q)}(B), \quad i = 1,2.$$

For the derivative of (3.6) with respect to B, we further obtain the following.

Lemma 3.2. *For every $x > B$,*

$$\frac{\partial}{\partial B}(e^B\Gamma^{(r+m)}(x-B)) = -\frac{\kappa(1)-(r+m)}{1-\Phi(r+m)}e^B\Theta^{(r+m)}(x-B).$$

Proof. By (3.5) and (3.7), the left-hand side equals

$$e^B(\Gamma^{(r+m)}(x-B) - \Gamma^{(r+m)\prime}(x-B)) = -e^B\left[\frac{\kappa(1)-(r+m)}{1-\Phi(r+m)}W^{(r+m)\prime}(x-B) \right.$$
$$\left. +(\kappa(1)-(r+m))W^{(r+m)}(x-B) - \frac{\kappa(1)-(r+m)}{1-\Phi(r+m)}W^{(r+m)}(x-B)\right],$$

which equals the right-hand side. □

By combining the two lemmas above, we obtain the derivative of $\mathcal{E}(x;B)$ with respect to B. For all $B \in \mathbb{R}$, define

$$J^{(r,m)}(B) := \left(\frac{r+m}{\Phi(r+m)} - \frac{r}{\Phi(r)}\right)\eta(B) - \left(H^{(r)}(B) - H^{(r+m)}(B)\right) \tag{3.8}$$

and

$$\begin{aligned} K_1^{(r,m)}(B) &:= \frac{\kappa(1)-(r+m)}{1-\Phi(r+m)}e^B - G_1^{(r+m)}(B) + G_2^{(r)}(B) - J^{(r,m)}(B), \\ K_2^{(r)}(B) &:= G_2^{(r)}(B) + \frac{r}{\Phi(r)}\eta(B) + H^{(r)}(B) + \frac{\sigma^2}{2}\eta'(B). \end{aligned} \tag{3.9}$$

Proposition 3.1. *For every $x > B$,*

$$\frac{\partial}{\partial B}\mathcal{E}(x;B) = -\left[\Theta^{(r+m)}(x-B)K_1^{(r,m)}(B) + \{\Theta^{(r)}(x-B) - \Theta^{(r+m)}(x-B)\}K_2^{(r)}(B)\right]. \tag{3.10}$$

Proof. Applying Lemmas 3.1-3.2 in (3.6),

$$\begin{aligned}\frac{\partial}{\partial B}\mathcal{E}(x;B) =& -\Theta^{(r+m)}(x-B)\frac{\kappa(1)-(r+m)}{1-\Phi(r+m)}e^B \\ &+\Theta^{(r+m)}(x-B)\left[G_1^{(r+m)}(B)+\frac{r+m}{\Phi(r+m)}\eta(B)+H^{(r+m)}(B)+\frac{\sigma^2}{2}\eta'(B)\right] \\ &-\Theta^{(r)}(x-B)\left[G_2^{(r)}(B)+\frac{r}{\Phi(r)}\eta(B)+H^{(r)}(B)+\frac{\sigma^2}{2}\eta'(B)\right] \\ =& -\Theta^{(r+m)}(x-B)\frac{\kappa(1)-(r+m)}{1-\Phi(r+m)}e^B \\ &+\Theta^{(r+m)}(x-B)\left[G_1^{(r+m)}(B)+\frac{r+m}{\Phi(r+m)}\eta(B)+H^{(r+m)}(B)+\frac{\sigma^2}{2}\eta'(B)\right] \\ &-\Theta^{(r+m)}(x-B)\left[G_2^{(r)}(B)+\frac{r}{\Phi(r)}\eta(B)+H^{(r)}(B)+\frac{\sigma^2}{2}\eta'(B)\right] \\ &-(\Theta^{(r)}(x-B)-\Theta^{(r+m)}(x-B))\left[G_2^{(r)}(B)+\frac{r}{\Phi(r)}\eta(B)+H^{(r)}(B)+\frac{\sigma^2}{2}\eta'(B)\right],\end{aligned}$$

which matches (3.10). □

Remark 3.3. *If $\eta(B)$ is increasing in B, then $K_2^{(r)}$ is uniformly positive.*

Remark 3.4. *We can also write*

$$J^{(r,m)}(B)=\frac{1}{2}\sigma^2(\Phi(r+m)-\Phi(r))\eta(B)+\int_0^\infty \Pi(\mathrm{d}u)\int_0^u (e^{-\Phi(r)z}-e^{-\Phi(r+m)z})\eta(B-u+z)\mathrm{d}z.$$

Consequently, because $\Phi(q)$ is increasing in q and $\eta(\cdot)$ is positive by assumption, $J^{(r+m)}(B)\geq 0$ for all $B\in\mathbb{R}$.

Proof. Because for any $q>0$

$$\frac{q}{\Phi(q)}=c+\frac{1}{2}\sigma^2\Phi(q)+\int_0^\infty \Pi(\mathrm{d}u)\left(\frac{e^{-\Phi(q)u}-1}{\Phi(q)}+u1_{\{u\in(0,1)\}}\right),$$

we obtain

$$\begin{aligned}\frac{r+m}{\Phi(r+m)}-\frac{r}{\Phi(r)} =& \frac{1}{2}\sigma^2\left(\Phi(r+m)-\Phi(r)\right)-\int_0^\infty \Pi(\mathrm{d}u)\left(\frac{e^{-\Phi(r)u}-1}{\Phi(r)}+u1_{\{u\in(0,1)\}}\right) \\ &+\int_0^\infty \Pi(\mathrm{d}u)\left(\frac{e^{-\Phi(r+m)u}-1}{\Phi(r+m)}+u1_{\{u\in(0,1)\}}\right) \\ =& \frac{1}{2}\sigma^2\left(\Phi(r+m)-\Phi(r)\right)+\int_0^\infty \Pi(\mathrm{d}u)\left(\frac{1-e^{-\Phi(r)u}}{\Phi(r)}-\frac{1-e^{-\Phi(r+m)u}}{\Phi(r+m)}\right).\end{aligned}$$

Substituting this in (3.8), we obtain the result.

□

3.4. **Continuous fit.** Before discussing the optimality, we consider the continuous fit condition:

$$\mathcal{E}(B+;B) = 0.$$

By taking $x \downarrow 0$ in (3.6),

(3.11)
$$\mathcal{E}(B+;B) = e^B \Gamma^{(r+m)}(0) + (\mathcal{M}_2^{(r)}(B+;B) - \Lambda^{(r)}(B+;B)) - (\mathcal{M}_1^{(r+m)}(B+;B) - \Lambda^{(r+m)}(B+;B)).$$

Here we have by (3.5)

$$\Gamma^{(r+m)}(0) = \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} W^{(r+m)}(0),$$

and by Proposition 3.2 of [7], for both $q = r$ and $q = r + m$,

$$\begin{aligned} \Lambda^{(q)}(B+;B) &= -W^{(q)}(0) \left(\frac{q}{\Phi(q)} \eta(B) + H^{(q)}(B) \right) + \eta(B), \\ \mathcal{M}_i^{(q)}(B+;B) &= W^{(q)}(0) G_i^{(q)}(B), \quad i = 1, 2. \end{aligned} \tag{3.12}$$

Substituting (3.12) in (3.11) and because $W^{(r)}(0) = W^{(r+m)}(0)$ as in (3.3), we obtain

$$\mathcal{E}(B+;B) = W^{(r+m)}(0) K_1^{(r,m)}(B). \tag{3.13}$$

By (3.3), we conclude that for the bounded variation case the continuous fit condition is equivalent to $K_1^{(r,m)}(B) = 0$, while for the unbounded variation case it always holds no matter how B is chosen.

Remark 3.5. *One can further pursue smooth fit for the case X is of unbounded variation. However, we do not discuss it here because it is not necessary for the proof of optimality in the subsequent sections. In fact, it can be expected that smooth fit condition $\mathcal{E}'(B+;B) = 0$ is equivalent to $K_1^{(r,m)}(B) = 0$, and the optimal solution is expected to satisfy smooth fit (at least when $\sigma > 0$ by the results obtained in* [7]*). Our numerical results in Section 5 verifies that this is indeed so.*

3.5. **Optimality.** We assume that there exists B^* such that

$$K_1^{(r,m)}(B) \geq 0 \Longleftrightarrow B \geq B^*, \tag{3.14}$$

$$K_2^{(r)}(B) \geq 0, \quad B \geq B^*, \tag{3.15}$$

and prove its optimality. We later discuss sufficient conditions that guarantee (3.14)-(3.15). Notice here that (3.15) always holds given $\eta(B)$ is increasing in view of Remark 3.3.

We first show via continuous fit and (3.14)-(3.15) that any feasible bankruptcy level must be at least as large as B^*. Toward this end, we use the following lemma.

Lemma 3.3. *When X is of unbounded variation, we have $\Theta^{(r)}(y) - \Theta^{(r+m)}(y) \to 0$ as $y \downarrow 0$.*

Proof. The result for the case $\sigma > 0$ is clear because, by (3.3), $W^{(q)}(0) = 0$ and $W^{(q)\prime}(0+) = 2/\sigma^2$ for any $q > 0$. Suppose $\sigma = 0$. Then as in the proof of Lemma 4.4 of [12], we have

$$\lim_{x\downarrow 0}\left[W^{(r)\prime}(x) - W^{(r+m)\prime}(x)\right] = \lim_{\lambda\uparrow\infty}\int_0^\infty \lambda e^{-\lambda x}\left[W^{(r)\prime}(x) - W^{(r+m)\prime}(x)\right]\mathrm{d}x$$
$$= \lim_{\lambda\uparrow\infty}\left[\frac{\lambda^2}{\kappa(\lambda)-r} - \frac{\lambda^2}{\kappa(\lambda)-(r+m)}\right] = -m\lim_{\lambda\uparrow\infty}\left[\frac{\lambda}{\kappa(\lambda)-r}\frac{\lambda}{\kappa(\lambda)-(r+m)}\right] = 0.$$

Here the last equality holds because for any $q > 0$

$$\frac{\kappa(\lambda)-q}{\lambda} = c + \int_{(0,\infty)}\left(\frac{e^{-\lambda x}-1}{\lambda} + x1_{\{0<x<1\}}\right)\Pi(\mathrm{d}x) - \frac{q}{\lambda} \xrightarrow{\lambda\uparrow\infty} \infty,$$

due to Fatou's lemma and $\int_{(0,1)} x\Pi(\mathrm{d}x) = \infty$. This together with $W^{(r)}(0) = W^{(r+m)}(0) = 0$ shows the result. □

Lemma 3.4. *Suppose there exists B^* such that (3.14) holds. If B satisfies (2.5), then $B \in [B^*, \infty)$.*

Proof. (i) Suppose X is of bounded variation and fix $\widehat{B} < B^*$. By (3.14), we have $K_1^{(r,m)}(\widehat{B}) < 0$. But by (3.13), this implies $\mathcal{E}(\widehat{B}+;\widehat{B}) < 0$, violating (2.5). Therefore, those B satisfying (2.5) must lie on $[B^*, \infty)$.

(ii) Suppose X is of unbounded variation and again fix $\widehat{B} < B^*$. For any sufficiently small $\delta > 0$ such that $K_1^{(r,m)}(x) < 0$ for every $\widehat{B} < x < \widehat{B} + \delta$, we have by Proposition 3.1

$$\inf_{\widehat{B}\le x\le \widehat{B}+\delta, \widehat{B}\le y\le x}\frac{\partial}{\partial B}\mathcal{E}(x;B)\Big|_{B=y} \ge \inf_{0\le y\le\delta}\Theta^{(r+m)}(y)\inf_{\widehat{B}\le B\le\widehat{B}+\delta}|K_1^{(r,m)}(B)|$$
$$- \sup_{0\le y\le \delta}\{\Theta^{(r)}(y) - \Theta^{(r+m)}(y)\}\sup_{\widehat{B}\le B\le\widehat{B}+\delta}|K_2^{(r)}(B)|.$$

This converges to some strictly positive value as $\delta \downarrow 0$ by Lemma 3.3, (3.3) and (3.7); namely there exists $\delta_0 > 0$ such that

$$\inf_{\widehat{B}\le x\le \widehat{B}+\delta_0, \widehat{B}\le y\le x}\frac{\partial}{\partial B}\mathcal{E}(x;B)\Big|_{B=y} > 0.$$

But by (3.13), $\mathcal{E}(\widehat{B}+\delta_0+;\widehat{B}+\delta_0) = 0$, implying $\mathcal{E}(\widehat{B}+\delta_0;\widehat{B}) < 0$, violating (2.5). Therefore the proof is complete by contradiction. □

Now, by how B^* is chosen, (3.14)-(3.15) and the positivity of both $\Theta^{(r)}(y) - \Theta^{(r+m)}(y)$ and $\Theta^{(r)}(y)$ for any $y > 0$, Proposition 3.1 implies

$$\frac{\partial}{\partial B}\mathcal{E}(x;B) < 0, \quad B^* \le B < x.$$

Moreover, B^* satisfies the liability constraint (2.5). Indeed, for any arbitrary $x > B^*$, we have by (3.13)

$$0 \le W^{(r+m)}(0)K_1^{(r,m)}(x) = \mathcal{E}(x+;x) < \mathcal{E}(x;B^*).$$

Here the first inequality holds because $K_1^{(r,m)}(x) \geq 0$ for any $x > B^*$ by (3.14) and in particular holds by equality for the unbounded variation case. This together with the lemma above shows the optimality of B^*. In summary, we have the following.

Theorem 3.1. *If there exists B^* such that (3.14)-(3.15) hold, then B^* is the optimal bankruptcy level.*

Remark 3.6. *The assumption that η is twice-differentiable on $\mathbb{R}$ as in Assumption 3.3 can be relaxed. Its twice-differentiability at a fixed $B \in \mathbb{R}$ is required for Lemma 3.1. In view of the arguments in this section (and in particular Remark 3.2), we only need it to hold Lebesgue-a.e. as long as η is continuous.*

4. Sufficient Conditions

In the last section, we showed that the conditions (3.14)-(3.15) guarantee the optimality of the bankruptcy level B^*. Here we obtain more concrete and economically sound conditions that satisfy (3.14)-(3.15). We first show that Assumption 4.1 below is sufficient and also encompasses the model by [8, 12] as reviewed in Section 2.1. We then give another example with constant η that does not satisfy Assumption 4.1 but nonetheless guarantees the optimality. It is emphasized here that the assumptions discussed in this section are sufficient and clearly not necessary; (3.14)-(3.15) are expected to hold more generally.

4.1. A sufficient condition.

We show that the following assumption guarantees (3.14)-(3.15) and hence the optimality of B^* holds by Theorem 3.1.

Assumption 4.1. *We suppose (1) η is increasing, (2) $\overline{\eta}$ is decreasing, (3) f_1 is decreasing, (4) f_2 is increasing, and (5) $0 \leq \overline{\eta}(\cdot) \leq 1$.*

Each condition in the assumption above has economic justifications. The monotonicity of η and $\overline{\eta}$ means that, as the firm's asset value increases, the amount of total bankruptcy costs increases and its proportion relative to the asset value decreases. Indeed, the total amount is naturally expected to be larger for larger firms and it is also known that its fraction tends to be smaller for large firms than small-and-medium-sized firms. The monotonicity of f_1 is justified by the fact that coupon rates tend to be smaller for the firms with better credit-ratings. The monotonicity of f_2 is valid (at least when the coupon rate is a fixed constant value) because as the bankruptcy becomes increasingly unlikely, the value of tax benefits increases to the capitalized value of tax benefits. The last condition requires that the bankruptcy costs should not be more than the total asset value.

We first note that (1) guarantees (3.15) by Remark 3.3. The following proposition shows that $K_1^{(r,m)}(B)$ is monotonically increasing and hence (3.14) also holds.

Proposition 4.1. *Suppose Assumption 4.1 holds. Then the unique root B^* of $K_1^{(r,m)}(B) = 0$ satisfies (3.14)-(3.15), and it is an optimal bankruptcy level.*

In order to prove Proposition 4.1, we first rewrite, in view of (3.9) and Remark 3.4,

$$K_1^{(r,m)}(B) = e^B l(B) - G_1^{(r+m)}(B) + G_2^{(r)}(B)$$

where

$$l(B) := \frac{\kappa(1)-(r+m)}{1-\Phi(r+m)} - \frac{1}{2}\sigma^2(\Phi(r+m)-\Phi(r))\overline{\eta}(B) - \int_0^\infty \Pi(\mathrm{d}u)\int_0^u (e^{-\Phi(r)z}-e^{-\Phi(r+m)z})e^{z-u}\overline{\eta}(B-u+z)\mathrm{d}z. \tag{4.1}$$

Because $\overline{\eta}(B)$ is decreasing in B by assumption and $\Phi(q)$ is increasing in q, $l(B)$ is increasing. Because $\overline{\eta}(B)$ is monotone and bounded in $[0,1]$, there exists $0 \leq \overline{\eta}(-\infty) := \lim_{B\downarrow-\infty}\overline{\eta}(B) \leq 1$. By the monotone convergence theorem, we obtain

$$\begin{aligned}\lim_{B\downarrow-\infty} l(B) &= \frac{\kappa(1)-(r+m)}{1-\Phi(r+m)} - \frac{1}{2}\sigma^2(\Phi(r+m)-\Phi(r))\overline{\eta}(-\infty) \\ &\quad - \int_0^\infty \Pi(\mathrm{d}u)\int_0^u (e^{-\Phi(r)z}-e^{-\Phi(r+m)z})e^{z-u}\overline{\eta}(-\infty)\mathrm{d}z \\ &= \frac{\kappa(1)-(r+m)}{1-\Phi(r+m)} - \overline{\eta}(-\infty)j^{(r,m)}\end{aligned}$$

where

$$j^{(r,m)} := \frac{1}{2}\sigma^2(\Phi(r+m)-\Phi(r)) + \int_0^\infty \Pi(\mathrm{d}u)e^{-u}\left(\frac{1-e^{-(\Phi(r)-1)u}}{\Phi(r)-1} - \frac{1-e^{-(\Phi(r+m)-1)u}}{\Phi(r+m)-1}\right).$$

Lemma 4.1. *We have*

$$j^{(r,m)} = \frac{\kappa(1)-(r+m)}{1-\Phi(r+m)} - \frac{\kappa(1)-r}{1-\Phi(r)}.$$

Proof. Define, as the Laplace exponent of X under $\mathbb{P}_1$ with the change of measure $\frac{\mathrm{d}\mathbb{P}_1}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{X_t-\kappa(1)t}$,

$$\kappa_1(\beta) := \left(\sigma^2 + c - \int_{(0,1)} u(e^{-u}-1)\Pi(\mathrm{d}u)\right)\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(0,\infty)}(e^{-\beta u}-1+\beta u 1_{\{u\in(0,1)\}})e^{-u}\,\Pi(\mathrm{d}u).$$

Then, $\kappa_1(\Phi(q)-1) = \kappa(\Phi(q)) - \kappa(1) = q - \kappa(1)$ for any $q > 0$; see page 215 of [10]. This shows

$$\begin{aligned}\frac{\kappa(1)-r}{1-\Phi(r)} = \frac{\kappa_1(\Phi(r)-1)}{\Phi(r)-1} &= \sigma^2 + c - \int_{(0,1)} u(e^{-u}-1)\Pi(\mathrm{d}u) \\ &\quad + \frac{1}{2}\sigma^2(\Phi(r)-1) + \int_{(0,\infty)}\left(\frac{e^{-(\Phi(r)-1)u}-1}{\Phi(r)-1} + u1_{\{u\in(0,1)\}}\right)e^{-u}\,\Pi(\mathrm{d}u)\end{aligned}$$

and

$$\begin{aligned}\frac{\kappa(1)-(r+m)}{1-\Phi(r+m)} = \frac{\kappa_1(\Phi(r+m)-1)}{\Phi(r+m)-1} &= \sigma^2 + c - \int_{(0,1)} u(e^{-u}-1)\Pi(\mathrm{d}u) \\ &\quad + \frac{1}{2}\sigma^2(\Phi(r+m)-1) + \int_{(0,\infty)}\left(\frac{e^{-(\Phi(r+m)-1)u}-1}{\Phi(r+m)-1} + u1_{\{u\in(0,1)\}}\right)e^{-u}\,\Pi(\mathrm{d}u).\end{aligned}$$

Subtracting the former from the latter, we obtain the result. □

Now by Lemma 4.1,

$$\lim_{B\downarrow -\infty} l(B) = (1-\overline{\eta}(-\infty))\frac{\kappa(1)-(r+m)}{1-\Phi(r+m)} + \overline{\eta}(-\infty)\frac{\kappa(1)-r}{1-\Phi(r)}.$$

Because $0 \leq \overline{\eta}(-\infty) \leq 1$ and κ is convex on $[0,\infty)$ and zero at the origin, we have $\lim_{B\downarrow -\infty} l(B) \geq 0$ and hence $l(B) \geq 0$ for any $B \in \mathbb{R}$. Consequently, $e^B l(B)$ is increasing in B. Finally, $-G_1^{(r+m)}(B) + G_2^{(r)}(B)$ is increasing in B because f_1 is decreasing and f_2 is increasing by assumption. Therefore, there exists a unique B^* that satisfies (3.14). As discussed above, (3.15) holds by the monotonicity of η. Now, by Theorem 3.1, Proposition 4.1 holds.

4.2. **Reduction to the case by [8, 12].** As an example that satisfies Assumption 4.1, we revisit the simple case by [8, 12] as reviewed in Section 2.1. Namely, we set

$$f_1(x) = P\hat{\rho} + p, \quad f_2(x) = 1_{\{x \geq \log V_T\}} P\hat{\gamma}\hat{\rho}, \quad \text{and} \quad \overline{\eta}(B) = \hat{\eta},$$

and confirm that our result matches that of [8, 12].

First, Assumption 4.1 is trivially satisfied and hence the optimal threshold level B^* is uniquely given by $K_1^{(r,m)}(B^*) = 0$. In this case,

$$G_1^{(r+m)}(B) = \int_0^\infty e^{-\Phi(r+m)y} f_1(y+B)\mathrm{d}y = \frac{P\hat{\rho}+p}{\Phi(r+m)} = \frac{P(\hat{\rho}+m)}{\Phi(r+m)} \tag{4.2}$$

and

$$\begin{aligned} G_2^{(r)}(B) &= \int_0^\infty e^{-\Phi(r)y} f_2(y+B)\mathrm{d}y = P\hat{\gamma}\hat{\rho}\int_{(\log V_T - B)\vee 0}^\infty e^{-\Phi(r)y}\mathrm{d}y = \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)} e^{-\Phi(r)((\log V_T - B)\vee 0)} \\ &= \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)}\left(e^{\log V_T - B} \vee 1\right)^{-\Phi(r)} = \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)}\left((e^{-B}V_T)\vee 1\right)^{-\Phi(r)} = \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)}\left(\frac{e^B}{V_T}\wedge 1\right)^{\Phi(r)}. \end{aligned}$$

We also have by Remark 3.4 and Lemma 4.1

$$\begin{aligned} J^{(r,m)}(B) &= \frac{1}{2}\sigma^2(\Phi(r+m)-\Phi(r))e^B\hat{\eta} + \int_0^\infty \Pi(\mathrm{d}u)\int_0^u (e^{-\Phi(r)z} - e^{-\Phi(r+m)z})e^{B-u+z}\hat{\eta}\mathrm{d}z \\ &= j^{(r,m)}\hat{\eta}e^B = \left(\frac{\kappa(1)-(r+m)}{1-\Phi(r+m)} - \frac{\kappa(1)-r}{1-\Phi(r)}\right)\hat{\eta}e^B. \end{aligned}$$

Combining the above,

$$\begin{aligned} K_1^{(r,m)}(B) &= \frac{\kappa(1)-(r+m)}{1-\Phi(r+m)}e^B - \frac{P(\hat{\rho}+m)}{\Phi(r+m)} + \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)}\left(\frac{e^B}{V_T}\wedge 1\right)^{\Phi(r)} - \hat{\eta}e^B\left(\frac{\kappa(1)-(r+m)}{1-\Phi(r+m)} - \frac{\kappa(1)-r}{1-\Phi(r)}\right) \\ &= -\frac{P(\hat{\rho}+m)}{\Phi(r+m)} + \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)}\left(\frac{e^B}{V_T}\wedge 1\right)^{\Phi(r)} + e^B\left((1-\hat{\eta})\frac{\kappa(1)-(r+m)}{1-\Phi(r+m)} + \hat{\eta}\frac{\kappa(1)-r}{1-\Phi(r)}\right). \end{aligned}$$

The unique value of B that satisfies $K_1^{(r,m)}(B) = 0$ indeed matches the result of [8, 12].

4.3. **Other examples.** Although Assumption 4.1 is economically a reasonable assumption, it is expected that (3.14)-(3.15) hold more generally. As an example Assumption 4.1 is violated but the optimality of B^* holds, we consider the case the value of bankruptcy costs is a constant, i.e., $\eta \equiv \eta_0$; see [14]. In this case, we have $\overline{\eta}(y) = \eta_0 e^{-y}$, which violates Assumption 4.1-(5). Nonetheless, the optimality trivially holds upon the monotonicity of $G_1^{(r+m)}$ and $G_2^{(r)}$. Indeed, $H^{(r)} \equiv H^{(r+m)} \equiv 0$ and hence, by (3.8), $J^{(r,m)}(B) = \left(\frac{r+m}{\Phi(r+m)} - \frac{r}{\Phi(r)}\right)\eta_0$, which is a constant. Now in view of the definition of $K_1^{(r+m)}$ in (3.9), it is clearly increasing in B for example when Assumption 4.1-(3,4) hold, and hence (3.14) is valid. Moreover, (3.15) trivially holds by Remark 3.3.

5. Numerical Examples

In this section, we illustrate how to compute the optimal bankruptcy level and the associated debt/equity/firm values. We also compute the optimal leverage ratio as a solution to the two-stage problem [5, 14, 15]. We use an example where Assumption 4.1 holds and, for X, we follow [8] and use a mixture of Brownian motion and a compound Poisson process with i.i.d. exponential jumps.

For the bankruptcy costs, let

$$\overline{\eta}(x) = \eta_0 \left(e^{-a(x-b)} \wedge 1\right), \quad x \in \mathbb{R},$$

for some $0 < a < 1$, $b \in \mathbb{R}$ and $0 \leq \eta_0 \leq 1$. This is clearly decreasing in x and bounded in $[0,1]$. Moreover,

$$\eta(x) = e^x \eta_0 \left(1 \wedge e^{-a(x-b)}\right) = \eta_0 \left(e^x \wedge e^{(1-a)x+ab}\right)$$

and

$$\eta'(x) = \begin{cases} \eta_0(1-a)e^{(1-a)x+ab}, & x > b, \\ \eta_0 e^x, & x < b, \end{cases}$$

and hence it is increasing. For the coupon rate we assume it is constant with $f_1 = P\hat{\rho} + p$ and for the tax benefits,

$$f_2(x) = P\hat{\gamma}\hat{\rho}\left(e^{x-c} \wedge 1\right). \tag{5.1}$$

for some $c \in \mathbb{R}$. Clearly, all the conditions in Assumption 4.1 are satisfied.

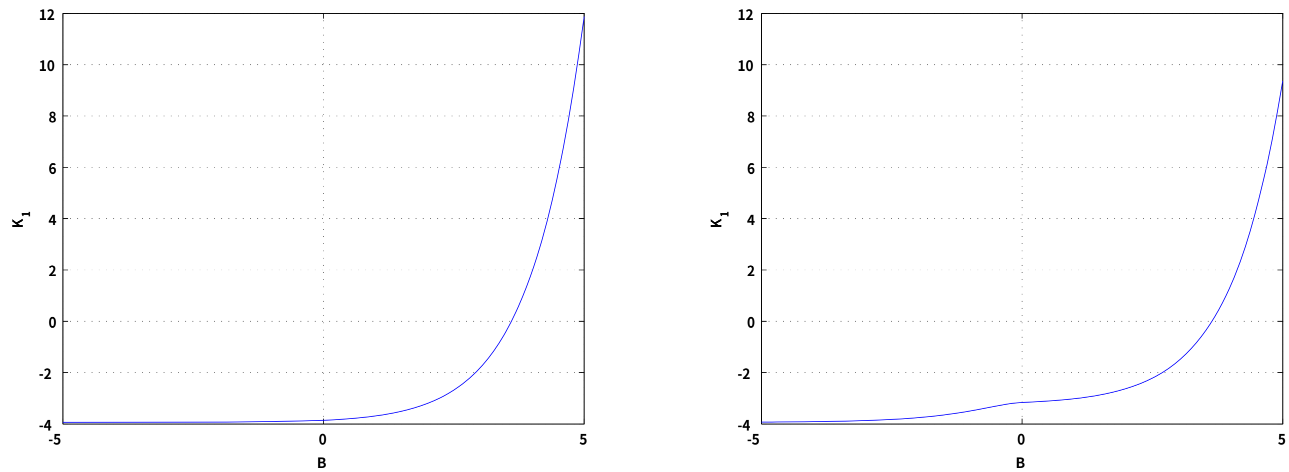
Regarding X, we consider the case $\sigma > 0$ and jumps are of exponential type with Lévy measure

$$\Pi(\mathrm{d}u) = \lambda\beta e^{-\beta u}\mathrm{d}u, \quad u > 0. \tag{5.2}$$

Its Laplace exponent (3.1) is given by

$$\kappa(s) = \mu s + \frac{1}{2}\sigma^2 s^2 + \lambda\left(\frac{\beta}{\beta+s} - 1\right), \quad s \in \mathbb{R}.$$

The scale function of this process has an explicit expression written in terms of a sum of exponential functions; see e.g. [6]. By straightforward but tedious algebra, we can obtain each functional in the equity value (3.6) as well as the function $K_1^{(r,m)}$ in (3.9) that determines the optimal bankruptcy level. For their explicit forms, see Appendix A.



case 1 case 2

FIGURE 1. The plots of $K_1^{(r,m)}(B)$. The unique root of $K_1^{(r,m)}(B) = 0$ becomes the optimal bankruptcy level B^*.

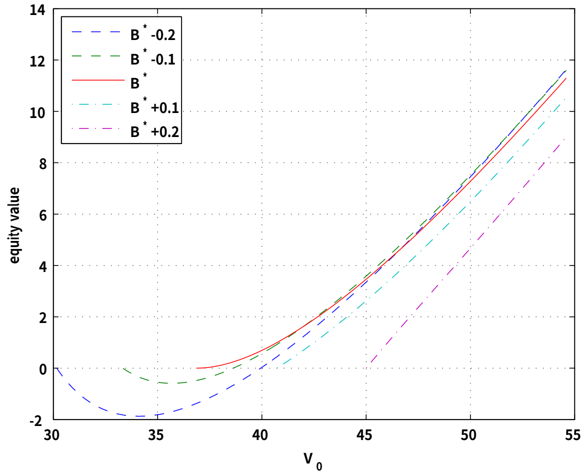
We use $r = 7.5\%$, $\delta = 7\%$, $\hat{\gamma} = 35\%$, $\sigma = 0.2$, $\lambda = 0.5$ and $\beta = 9$ which were used in [8, 12, 14, 15]. We also use $\hat{\rho} = 8.162\%$ and $m = 0.2$, which were used in [5]. We choose the drift term μ so that the martingale property $\kappa(1) = r - \delta$ is satisfied. Regarding the parameters for η, f_1 and f_2 defined above, we consider the following two cases:

case 1: $\eta_0 = 0.9$, $a = 0.5$, $b = 0$ and $c = 5$,
case 2: $\eta_0 = 0.5$, $a = 0.01$, $b = 5$ and $c = 0$.

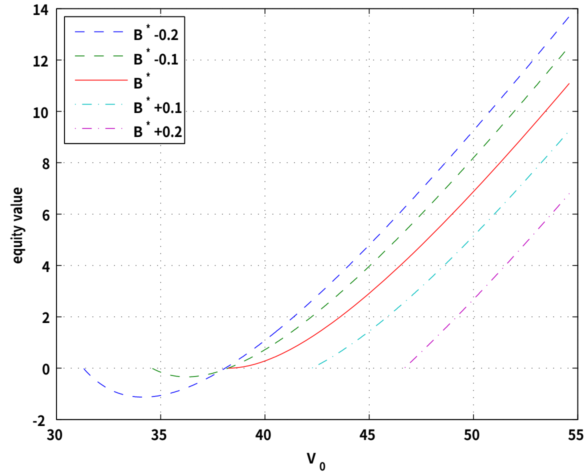
In case 1, the slopes of $\overline{\eta}$ and f_2 are magnified by how the parameters are chosen. On the other hand, in case 2, these values are constant at least when $x \in [0, 5]$, making the model similar to [8, 12].

Figure 1 shows the function $K_1^{(r+m)}$ in (3.9) as a function of B when $P = 50$. As shown in the proof of Proposition 4.1, this is indeed monotonically increasing for both cases. The optimal bankruptcy level B^* can therefore be computed by the bisection method and we obtain $B^* = 3.61$ and $B^* = 3.64$ for cases 1 and 2, respectively. In case 2, we notice a non-smooth point at zero and this is caused because $c = 0$ in the definition of f_2 in (5.1). However, because B^* is chosen larger than zero, the tax rate becomes constant until bankruptcy. Furthermore, because $B^* < b$, the loss fraction $\overline{\eta}$ at bankruptcy ends up being a constant. In case 1, on the other hand, the tax rate fluctuates over time and $\overline{\eta}$ is also level-dependent.

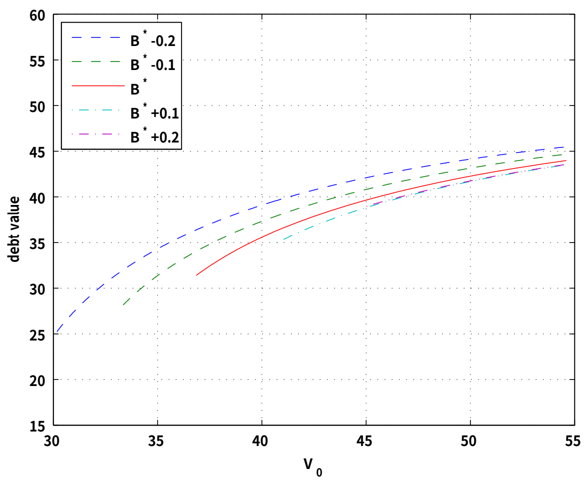
Using the optimal bankruptcy levels B^* computed above, we can compute the debt/equity/firm values. In Figure 2, we plot their values as a function of $V_0 = e^x \geq B$ for $B = B^* - 0.2, B^* - 0.1, B^*, B^* + 0.1, B^* + 0.2$. As shown in Lemma 3.4, we can confirm that, when B is taken lower than B^*, it violates



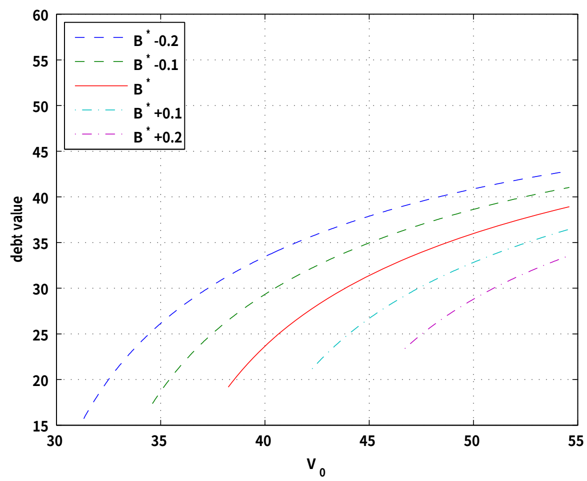
equity value (case 1)



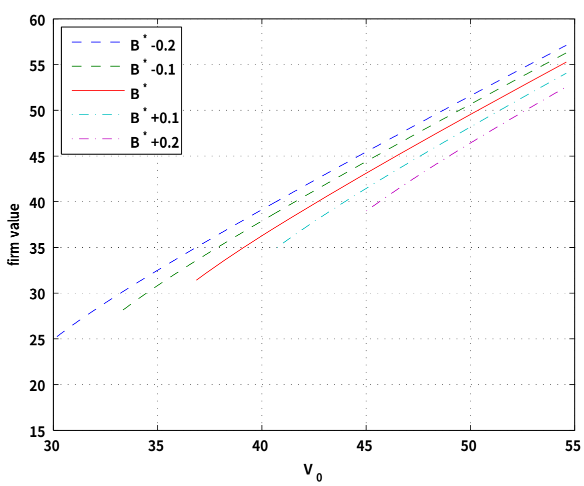
equity value (case 2)



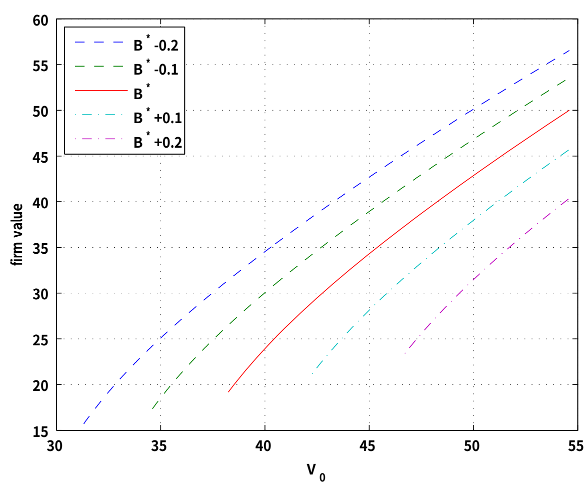
debt value (case 1)



debt value (case 2)



firm value (case 1)



firm value (case 2)

FIGURE 2. The equity/debt/firm values as a function of V_0 for various values of B.

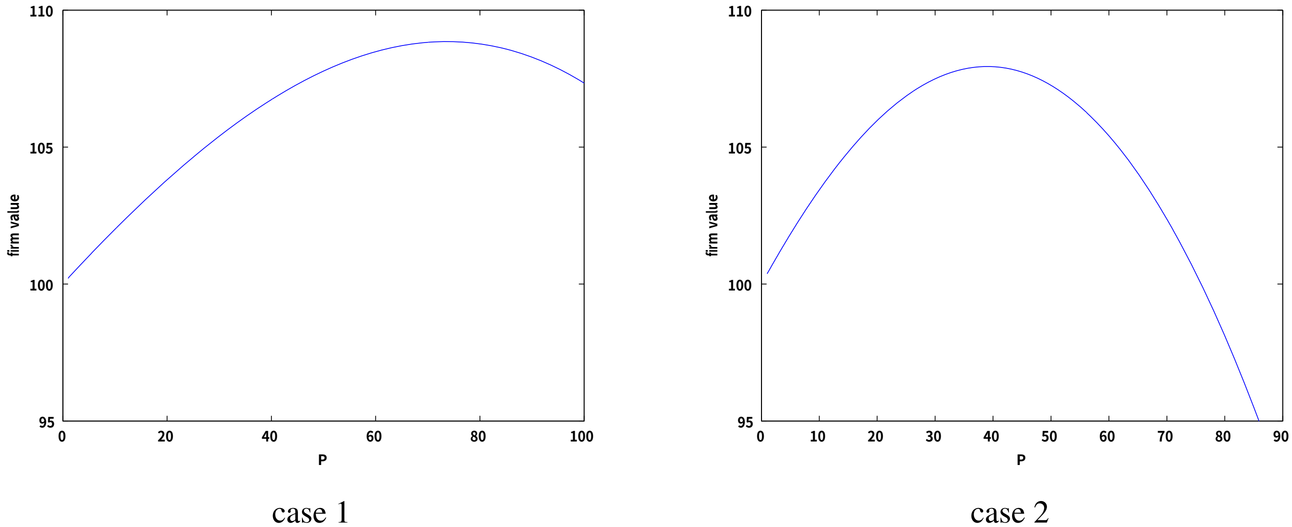


FIGURE 3. The firm value as a function of P for the two-stage problem.

the limit liability constraint (2.5). For B larger than B^*, the equity value is dominated by the value under B^*. We note that continuous fit at B always holds as in (3.13). Also, as we have discussed in Remark 3.5, we also observe that smooth fit holds at B^*.

We further proceed to solve the *two-stage problem* as studied by [5, 14, 15] where the final goal is to choose P such that the firm's value $\mathcal{V}$ is maximized, namely, for fixed x,

$$\max_P \mathcal{V}(x; B^*(P), P)$$

where we emphasize the dependency of $\mathcal{V}$ and B^* on P. We set $V_0 = 100$ (or $x = \log(100)$) and obtain B^* for P running from 0 to 100. The corresponding firm value $\mathcal{V}$ is computed for each P and $B^* = B^*(P)$, and is shown in Figure 3. As can be seen easily, these are concave functions and hence the optimal face values of debt $P^* = 73.7$ and $P^* = 39$ are obtained for cases 1 and 2, respectively.

The computation we conducted here is efficient and does not rely on heavy algorithms. This is partly due to the form of its scale function (see Appendix A) and also to the choice of η, f_1 and f_2 given above. However, this can be extended very easily to the hyperexponential case; see [6]. For other spectrally negative Lévy processes with explicit forms of scale functions, see [9, 11, 12]. We also remark that the solutions can in principle be computed numerically for any choice of spectrally negative Lévy process by using the approximation algorithms of the scale function such as [6, 16].

6. CONCLUDING REMARKS

We have studied a generalization of the Leland-Toft optimal capital structure model where the values of bankruptcy costs, coupon payments and tax benefits are dependent on the asset value. Focusing on the spectrally negative Lévy model, we obtained a sufficient condition for optimality, which holds under some economically reasonable assumptions. The solutions admit semi-explicit forms in terms of the scale function and allow for instant computation of the optimal capital structure. The generalization we achieved in this paper can be applied to realize more flexible models to derive optimal capital structures.

For future research, it is most natural to consider its extension to a general Lévy case with two-sided jumps. It is expected that the same generalization can be achieved at least for the Lévy processes admitting rational forms of Wiener-Hopf factorization such as double exponential jump diffusion [5] and, more generally, the phase-type Lévy process [1]. It is also beneficial to obtain other sufficient conditions for optimality that we have not covered in Section 4. Finally, a model has to adapt to a changing economic environment and financial restrictions. An interesting but nonetheless challenging extension would be to change the limit liability constraint so that the equity value must be bounded from below by some positive constant as opposed to be kept simply above zero as in the current model.

Appendix A. The Equity/Debt/Firm values for Section 5

In this appendix, we obtain explicit expressions of the function $K_1^{(r,m)}$ as well as the equity/debt/firm values that are used in Section 5.

First, the scale function for the process with $\sigma > 0$ and Lévy measure (5.2) admits an explicit form. For every $q > 0$, there are two negative real roots $-\xi_{1,q}$ and $-\xi_{2,q}$ to the equation $\kappa(s) = q$ and, as is discussed in [6], its scale function is given by

$$\begin{aligned} W^{(q)}(x) &= \sum_{i=1,2} C_{i,q}\left[e^{\Phi(q)x} - e^{-\xi_{i,q}x}\right], \\ Z^{(r)}(x) &= 1 + r\sum_{i=1,2} C_{i,q}\left[\frac{1}{\Phi(r)}\left(e^{\Phi(r)x}-1\right) + \frac{1}{\xi_{i,r}}\left(e^{-\xi_{i,r}x}-1\right)\right], \end{aligned} \tag{A.1}$$

for some positive constants $C_{1,q}$ and $C_{2,q}$; see [6] for their explicit expressions.

For the computation of $K_1^{(r,m)}$ in (3.9), straight algebra obtains, for every $B \in \mathbb{R}$ and $q > 0$,

(A.2)
$$G_1^{(q)}(B) = \frac{P(\hat{\rho}+m)}{\Phi(q)} \quad \text{and} \quad G_2^{(q)}(B) = P\hat{\gamma}\hat{\rho}\left[e^{B-c}\frac{1-e^{-(\Phi(q)-1)[(c-B)\vee 0]}}{\Phi(q)-1} + \frac{1}{\Phi(q)}e^{-\Phi(q)[(c-B)\vee 0]}\right],$$

and

$$H^{(q)}(B) = \lambda\eta(B)\left(\frac{1}{\Phi(q)+\beta}\right) - e^B Q(B;\Phi(q),\infty), \tag{A.3}$$

where

$$Q(B;\zeta,l) := \int_0^\infty \Pi(\mathrm{d}u)\int_0^{u\wedge l} e^{-(\zeta-1)z-u}\overline{\eta}(B-u+z)\mathrm{d}z, \quad \zeta,\ l>0.$$

Here, it can be shown, for any $\zeta \in \mathbb{R}$ and $l > 0$ with $\widetilde{b} = (B-b) \vee 0$,

$$\begin{aligned} Q(B;\zeta,l)/\eta_0 &= \frac{\lambda\beta}{\zeta-1}\Bigg[\left(\frac{e^{-\widetilde{b}(1+\beta)}}{1+\beta}\left(1-e^{-l(1+\beta)}\right) - \frac{e^{-\widetilde{b}(\zeta+\beta)+(\zeta-1)\widetilde{b}}}{\zeta+\beta}\left(1-e^{-l(\zeta+\beta)}\right)\right) \\ &\qquad + \frac{1}{1+\beta}e^{-(l+\widetilde{b})(1+\beta)}\left(1-e^{-(\zeta-1)l}\right)\Bigg] \\ &+ \frac{\lambda\beta}{\zeta-1+a}\Bigg[\frac{1}{\beta+1-a}\left(e^{-a\widetilde{b}} - e^{-\widetilde{b}(\beta+1)}\right) + \frac{1}{\zeta+\beta}\left(e^{-\widetilde{b}(\beta+1)} - e^{-(\widetilde{b}+l)(\zeta+\beta)+\widetilde{b}(\zeta-1)}\right) \\ &\qquad + \frac{1}{\beta+1-a}e^{-l(\beta+\zeta)-\widetilde{b}(\beta+1-a)-a\widetilde{b}}\Bigg] \\ &- \frac{\lambda\beta e^{-a\widetilde{b}}}{\zeta-1+a}\left[\frac{1}{\zeta+\beta}\left(1-e^{-l(\zeta+\beta)}\right) + \frac{1}{1-a+\beta}e^{-l(\zeta+\beta)}\right]. \end{aligned}$$

For the equity/debt/firm values, we need $\mathcal{M}_i^{(q)}(x;B)$ and $\Lambda^{(q)}(x;B)$ in (3.4). For the former, for every $q > 0$ and $x > B$,

$$\int_B^x W^{(q)}(x-y)f_1(y)\mathrm{d}y = (P\hat{\rho}+p)\sum_{i=1,2} C_{i,q}\left[\frac{1}{\Phi(q)}\left(e^{\Phi(q)(x-B)}-1\right) - \frac{1}{\xi_{i,q}}\left(1-e^{-\xi_{i,q}(x-B)}\right)\right]$$

and

$$\begin{aligned} &\int_B^x W^{(q)}(x-y)f_2(y)\mathrm{d}y = P\hat{\rho}\gamma\sum_{i\in 1,2} C_{i,q} \\ &\times\Bigg[e^{-c}\left[\frac{e^{\Phi(q)x}}{\Phi(q)-1}\left(e^{-(\Phi(q)-1)B} - e^{-(\Phi(q)-1)(x\wedge c\vee B)}\right) - \frac{e^{-\xi_{i,q}x}}{\xi_{i,q}+1}\left(e^{(\xi_{i,q}+1)(x\wedge c\vee B)} - e^{(\xi_{i,q}+1)B}\right)\right] \\ &\qquad + \left[\frac{1}{\Phi(q)}\left(e^{\Phi(q)(x-x\wedge c\vee B)}-1\right) - \frac{1}{\xi_{i,q}}\left(1-e^{-\xi_{i,q}(x-x\wedge c\vee B)}\right)\right]\Bigg]. \end{aligned}$$

These together with (A.1)-(A.2) obtain $\mathcal{M}_i^{(q)}(x;B)$ for any $i = 1,2$ and $x > B$.

For $\Lambda^{(q)}(x;B)$, some algebra shows

$$\begin{aligned} &\int_0^\infty \Pi(\mathrm{d}u)\int_0^{u\wedge(x-B)} W^{(q)}(x-z-B)\mathrm{d}z \\ &\qquad = \lambda\sum_{i=1,2} C_{i,q}\left[\frac{1}{\Phi(q)+\beta}\left(e^{\Phi(q)(x-B)} - e^{-\beta(x-B)}\right) + \frac{1}{\xi_{i,q}-\beta}\left(e^{-\xi_{i,q}(x-B)} - e^{-\beta(x-B)}\right)\right] \end{aligned}$$

and

$$\begin{aligned} &\int_0^\infty \Pi(\mathrm{d}u)\left[\int_0^{u\wedge(x-B)} W^{(q)}(x-z-B)\eta(z+B-u)\mathrm{d}z\right] \\ &\qquad = \sum_{i=1,2} C_{i,q}e^{\Phi(q)(x-B)+B}Q(B;\Phi(q),x-B) - \sum_{i=1,2} C_{i,q}e^{-\xi_{i,q}(x-B)+B}Q(B;-\xi_{i,q},x-B). \end{aligned}$$

Now $\Lambda^{(q)}(x; B)$ is immediately obtained by (A.1) and (A.3).

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