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DEFAULT SWAP GAMES DRIVEN BY SPECTRALLY NEGATIVE LÉVY PROCESSES*

MASAHIKO EGAMI[◇], TIM S.T. LEUNG[†], AND KAZUTOSHI YAMAZAKI[‡]

ABSTRACT. This paper studies the valuation of game-type credit default swaps (CDSs) that allow the protection buyer and seller to raise or reduce the respective position once prior to default. This leads to the study of a stochastic game with optimal stopping subject to early termination resulting from a default. Under a structural credit risk model based on spectrally negative Lévy processes, we analyze the existence of the Nash equilibrium and derive the associated saddle point. Using the principles of smooth and continuous fit, we determine the buyer's and seller's equilibrium exercise strategies, which are of threshold type. Numerical examples are provided to illustrate the impacts of default risk and contractual features on the fair premium and exercise strategies.

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1. INTRODUCTION

Credit default swaps (CDSs) are among the most liquid and widely used credit derivatives for trading and managing default risks. Under a vanilla CDS contract, the protection buyer pays a periodic premium to the protection seller in exchange for a payment if the reference entity defaults before expiration. In a recent work [27], we have studied the step-up and step-down CDSs which provide the buyer or the seller the timing option to adjust the premium and notional amount once prior to default. These contracts give the investor valuable flexibility to control the credit risk exposure, and generalize the common callable and putable CDSs.

The current paper studies the valuation of game-type CDSs that allow both the protection buyer and seller to change the swap position once prior to default. Specifically, in the step-up (resp. step-down) default swap game, as soon as the buyer or the seller, whoever first, exercises prior to default, the notional amount and premium will be increased (resp. decreased) to a pre-specified level upon exercise. From the exercise time till default, the buyer will pay the new premium and the seller is subject to the new default liability. Hence, for a given set of contract parameters, the buyer's objective is to maximize the expected net cash flow while the seller wants to minimize it, giving rise to a two-player optimal stopping game.

We model the default time as the *first passage time* of a general *exponential Lévy process* representing some underlying asset value. This is an extension of the original structural credit risk approach introduced

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by Black and Cox [9] where the asset value follows a geometric Brownian motion. Examples of other structural models based on Lévy and other jump processes include [10, 19, 36].

The default swap game is formulated as a variation of the standard optimal stopping games in the literature (see, among others, [13, 17] and references therein). However, while typical optimal stopping games end at the time of exercise by either player, the exercise time in the default swap game does not terminate the contract, but merely alters the premium forward and the future protection amount to be paid at default time. In fact, since default may arrive before either party exercises, the game may be terminated early involuntarily.

The central challenge of the default swap games lies in determining the pair of stopping times that yield the *Nash equilibrium*. Under a structural credit risk model based on *spectrally negative* Lévy processes, we analyze and calculate the equilibrium exercise strategies for the protection buyer and seller. In addition, we determine the fair premium of the default swap game so that the expected discounted cash flows for the two parties coincide at contract inception.

Our solution approach starts with a decomposition of the default swap game into a combination of a perpetual CDS and an optimal stopping game with early termination from default. Moreover, we utilize a symmetry between the step-up and step-down games, which significantly simplifies our analysis as it is sufficient to study either case. For spectrally negative Lévy processes with a completely monotone Lévy density, we provide the conditions for the existence of the Nash equilibrium. Moreover, we derive the buyer's and seller's optimal threshold-type exercise strategies using the principle of continuous and smooth fit, followed by a rigorous verification theorem via martingale arguments.

For our analysis of the game equilibrium, the *scale function* and a number of fluctuation identities of spectrally negative Lévy processes are particularly useful. Using our analytic results, we provide a bisection-based algorithm for the efficient computation of the buyer's and seller's exercise thresholds as well as the fair premium, illustrated in a series of numerical examples. Other recent applications of spectrally negative Lévy processes include derivative pricing [1, 3], optimal dividend problem [4, 24, 29], and capital reinforcement timing [16]. We refer the reader to [23] for a comprehensive account.

To our best knowledge, the step-up and step-down default swap games and the associated optimal stopping games have not been studied elsewhere. There are a few related studies on stochastic games driven by spectrally negative or positive Lévy processes; see e.g. [6] and [7]. For optimal stopping games driven by a strong Markov process, we refer to the recent papers by [17] and [34], which study the existence and mathematical characterization of Nash and Stackelberg equilibria. Other game-type derivatives in the literature include Israeli/game options [21, 22], defaultable game options [8], and convertible bonds [20, 35].

The rest of the paper is organized as follows. In Section 2, we formulate the default swap game valuation problems under a general Lévy model. In Section 3, we focus on the spectrally negative Lévy model and provide a complete solution and detailed analysis. Section 4 provides the numerical study of the default swap games and a discussion on numerical approximation of scale functions for implementation. Section 5 concludes the paper and presents some ideas for future work. All proofs, unless otherwise noted, are given in the Appendix.

2. GAME FORMULATION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where $\mathbb{P}$ is the *risk-neutral* measure used for pricing. We assume there exists a Lévy process $X = \{X_t;\, t \geq 0\}$, and denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by X. The value of the reference entity (a company stock or other assets) is assumed to evolve according to an *exponential Lévy process* $S_t = e^{X_t}$, $t \geq 0$. Following the Black-Cox [9] structural approach, the default event is triggered by S crossing a lower level D, so the default time is given by the first passage time: $\theta_D := \inf\{\, t \geq 0 \,:\, X_t \,\leq\, \log D \,\}$. Without loss of generality, we can take $\log D = 0$ by shifting the initial value x. Henceforth, we shall work with the default time:

$$\theta := \inf\{\, t \geq 0 \,:\, X_t \,\leq\, 0 \,\},$$

where we assume $\inf \emptyset = \infty$. We denote by $\mathbb{P}^x$ the probability law and $\mathbb{E}^x$ the expectation under which $X_0 = x \in \mathbb{R}$.

We consider a game where the premium rate and default payment are changed from p to $\hat{p}$ and α to $\hat{\alpha}$ at the time the buyer or the seller exercises whichever comes first, provided that it is *strictly before* default. When the buyer exercises, she is incurred the fee γ_b to be paid to the seller; when the seller exercises, she is incurred γ_s to be paid to the buyer. If the buyer and the seller exercise simultaneously, then both parties pay the fee upon exercise. We assume that p, $\hat{p}$, α, $\hat{\alpha}$, γ_b, $\gamma_s \geq 0$ (see also Remark 2.1 below).

Let $\mathcal{S} := \{\tau \in \mathbb{F} : \tau \leq \theta \text{ a.s. }\}$ be the set of all stopping times *smaller than or equal to* the default time. Denote the buyer's candidate exercise time by $\tau \in \mathcal{S}$ and seller's candidate exercise time by $\sigma \in \mathcal{S}$, and let $r > 0$ be the positive risk-free interest rate. Given any pair of exercise times (σ, τ), the expected cash flow to the buyer is given by

$$\begin{aligned} V(x;\sigma,\tau) := \mathbb{E}^x \Bigg[-\int_0^{\tau\wedge\sigma} e^{-rt} p \,\mathrm{d}t + 1_{\{\tau\wedge\sigma<\infty\}} \Bigg(-\int_{\tau\wedge\sigma}^{\theta} e^{-rt} \hat{p} \,\mathrm{d}t \\ + e^{-r\theta} (\hat{\alpha} 1_{\{\tau\wedge\sigma<\theta\}} + \alpha 1_{\{\tau\wedge\sigma=\theta\}}) + 1_{\{\tau\wedge\sigma<\theta\}} e^{-r(\tau\wedge\sigma)} \left(-\gamma_b 1_{\{\tau\leq\sigma\}} + \gamma_s 1_{\{\tau\geq\sigma\}} \right) \Bigg) \Bigg]. \end{aligned} \tag{2.1}$$

To the seller, the contract value is $-V(x;\sigma,\tau)$. Naturally, the buyer wants to *maximize* V over τ whereas the seller wants to *minimize* V over σ, giving rise to a two-player optimal stopping game.

This formulation covers default swap games with the following provisions:

(1) *Step-up Game*: if $\hat{p} > p$ and $\hat{\alpha} > \alpha$, then the buyer and the seller are allowed to *increase* the notional amount once from α to $\hat{\alpha}$ and the premium rate from p to $\hat{p}$ by paying the fee γ_b (if the buyer exercises) or γ_s (if the seller exercises).

(2) *Step-down Game*: if $\hat{p} < p$ and $\hat{\alpha} < \alpha$, then the buyer and the seller are allowed to *decrease* the notional amount once from α to $\hat{\alpha}$ and the premium rate from p to $\hat{p}$ by paying the fee γ_b (if the buyer exercises) or γ_s (if the seller exercises). When $\hat{p} = \hat{\alpha} = 0$, we obtain a *cancellation game* which allows the buyer and the seller to terminate the contract early.

Our primary objective is to find a *Nash equilibrium* and *Stackelberg equilibrium*. A Nash equilibrium means the existence of a *saddle point* (σ^*, τ^*) such that

$$V(x;\sigma^*,\tau) \leq V(x;\sigma^*,\tau^*) \leq V(x;\sigma,\tau^*), \quad \forall\, \tau,\sigma \in \mathcal{S}. \tag{2.2}$$

A Stackelberg equilibrium means that $V^*(x) = V_*(x)\ \forall x \in \mathbb{R}$, where

$$V^*(x) := \inf_{\sigma\in\mathcal{S}} \sup_{\tau\in\mathcal{S}} V(x;\sigma,\tau) \quad \text{and} \quad V_*(x) := \sup_{\tau\in\mathcal{S}} \inf_{\sigma\in\mathcal{S}} V(x;\sigma,\tau).$$

See e.g. [17] and [34]. It follows from these definitions that $V^*(x) \geq V_*(x)$. If a Nash equilibrium is attained, then (2.2) implies the reverse inequality:

$$V^*(x) \leq \sup_{\tau\in\mathcal{S}} V(x;\sigma^*,\tau) \leq V(x;\sigma^*,\tau^*) \leq \inf_{\sigma\in\mathcal{S}} V(x;\sigma,\tau^*) \leq V_*(x).$$

Hence, a Nash equilibrium implies a Stackelberg equilibrium. Moreover, we also seek to determine the equilibrium premium $p^*(x)$ so that $V(x;\sigma^*,\tau^*) = 0$, yielding no cash transaction at contract initiation.

2.1. **Decomposition and Symmetry.** We begin our analysis with two useful observations, namely, the decomposition of V and the symmetry between the step-up and step-down games. In standard optimal stopping games, such as the well-known Dynkin game [13], random payoffs are realized at either player's exercise time. However, our default swap game is not terminated at the buyer's or seller's exercise time. In fact, upon exercise only the contract terms will change, and there will be a terminal transaction at default time. Since default may arrive before either party exercises the step-up/down option, the game may be terminated early involuntarily. Therefore, we seek to transform the value function V into another optimal stopping game that is more amenable for analysis.

First, we define the value of a (perpetual) CDS with premium rate p and notional amount α by

$$C(x;p,\alpha) := \mathbb{E}^x\left[-\int_0^\theta e^{-rt}p\,\mathrm{d}t + \alpha\, e^{-r\theta}\right] = \left(\frac{p}{r}+\alpha\right)\zeta(x) - \frac{p}{r}, \quad x\in\mathbb{R}, \tag{2.3}$$

where

$$\zeta(x) := \mathbb{E}^x\left[e^{-r\theta}\right], \quad x\in\mathbb{R}, \tag{2.4}$$

is the Laplace transform of θ. Next, we extract this CDS value from the value function V. Let

$$\tilde{\alpha} := \alpha - \hat{\alpha} \quad \text{and} \quad \tilde{p} := p - \hat{p}. \tag{2.5}$$

Proposition 2.1 (decomposition). *For every $\sigma,\tau\in\mathcal{S}$ and $x\in\mathbb{R}$, the value function admits the decomposition*

$$V(x;\sigma,\tau) = C(x;p,\alpha) + v(x;\sigma,\tau),$$

where $v(x;\sigma,\tau) \equiv v(x;\sigma,\tau;\tilde{p},\tilde{\alpha},\gamma_b,\gamma_s)$ is defined by

$$v(x;\sigma,\tau;\tilde{p},\tilde{\alpha},\gamma_b,\gamma_s) := \mathbb{E}^x\left[e^{-r(\tau\wedge\sigma)}\left(h(X_\tau)1_{\{\tau<\sigma\}} + g(X_\sigma)1_{\{\tau>\sigma\}} + f(X_\tau)1_{\{\tau=\sigma\}}\right)1_{\{\tau\wedge\sigma<\infty\}}\right], \tag{2.6}$$

with

$$h(x) \equiv h(x;\tilde{p},\tilde{\alpha},\gamma_b) := 1_{\{x>0\}}\left[\left(\frac{\tilde{p}}{r}-\gamma_b\right) - \left(\frac{\tilde{p}}{r}+\tilde{\alpha}\right)\zeta(x)\right], \tag{2.7}$$

$$g(x) \equiv g(x;\tilde{p},\tilde{\alpha},\gamma_s) := 1_{\{x>0\}}\left[\left(\frac{\tilde{p}}{r}+\gamma_s\right) - \left(\frac{\tilde{p}}{r}+\tilde{\alpha}\right)\zeta(x)\right], \tag{2.8}$$

$$f(x) \equiv f(x;\tilde{p},\tilde{\alpha},\gamma_b,\gamma_s) := 1_{\{x>0\}}\left[\left(\frac{\tilde{p}}{r}-\gamma_b+\gamma_s\right) - \left(\frac{\tilde{p}}{r}+\tilde{\alpha}\right)\zeta(x)\right]. \tag{2.9}$$

Proof. First, by a rearrangement of integrals and (2.5), the expression inside the expectation in (2.1) can be written as

$$1_{\{\tau\wedge\sigma<\infty\}}\left(\int_{\tau\wedge\sigma}^{\theta} e^{-rt}\tilde{p}\,\mathrm{d}t - \int_0^{\theta} e^{-rt}p\,\mathrm{d}t + e^{-r\theta}\left(-\tilde{\alpha}1_{\{\tau\wedge\sigma<\theta\}}+\alpha\right) + 1_{\{\tau\wedge\sigma<\theta\}}e^{-r(\tau\wedge\sigma)}\left(-\gamma_b 1_{\{\tau\le\sigma\}}+\gamma_s 1_{\{\tau\ge\sigma\}}\right)\right)$$
$$+ 1_{\{\tau\wedge\sigma=\infty\}}\left(-\int_0^{\infty} e^{-rt}p\,\mathrm{d}t\right)$$
$$= 1_{\{\tau\wedge\sigma<\infty\}}\left(\int_{\tau\wedge\sigma}^{\theta} e^{-rt}\tilde{p}\,\mathrm{d}t - e^{-r\theta}\tilde{\alpha}1_{\{\tau\wedge\sigma<\theta\}} + 1_{\{\tau\wedge\sigma<\theta\}}e^{-r(\tau\wedge\sigma)}\left(-\gamma_b 1_{\{\tau\le\sigma\}}+\gamma_s 1_{\{\tau\ge\sigma\}}\right)\right)$$
$$-\int_0^{\theta} e^{-rt}p\,\mathrm{d}t + e^{-r\theta}\alpha$$
$$= 1_{\{\tau\wedge\sigma<\infty,\,\tau\wedge\sigma<\theta\}}\left(\int_{\tau\wedge\sigma}^{\theta} e^{-rt}\tilde{p}\,\mathrm{d}t - e^{-r\theta}\tilde{\alpha} + e^{-r(\tau\wedge\sigma)}\left(-\gamma_b 1_{\{\tau\le\sigma\}}+\gamma_s 1_{\{\tau\ge\sigma\}}\right)\right) - \int_0^{\theta} e^{-rt}p\,\mathrm{d}t + e^{-r\theta}\alpha.$$

Taking expectation, (2.1) simplifies to

$$V(x;\sigma,\tau) = \mathbb{E}^x\left[1_{\{\tau\wedge\sigma<\infty,\,\tau\wedge\sigma<\theta\}}\left(\int_{\tau\wedge\sigma}^{\theta} e^{-rt}\tilde{p}\,\mathrm{d}t - e^{-r\theta}\tilde{\alpha} + e^{-r(\tau\wedge\sigma)}\left(-\gamma_b 1_{\{\tau\le\sigma\}}+\gamma_s 1_{\{\tau\ge\sigma\}}\right)\right)\right] \tag{2.10}$$
$$-\mathbb{E}^x\left[\int_0^{\theta} e^{-rt}p\,\mathrm{d}t\right] + \alpha\,\mathbb{E}^x\left[e^{-r\theta}\right].$$

Here, the last two terms do not depend on τ nor σ and they constitute $C(x;p,\alpha)$. Next, using the fact that $\{\tau\wedge\sigma<\theta, \tau\wedge\sigma<\infty\} = \{X_{\tau\wedge\sigma}>0,\ \tau\wedge\sigma<\infty\}$ for every $\tau,\sigma\in\mathcal{S}$ and the strong Markov property of X at time $\tau\wedge\sigma$, we express the first term as

$$\mathbb{E}^x\left[1_{\{\tau\wedge\sigma<\infty,\,\tau\wedge\sigma<\theta\}}\left(\mathbb{E}^x\left[\int_{\tau\wedge\sigma}^{\theta} e^{-rt}\tilde{p}\,\mathrm{d}t - e^{-r\theta}\tilde{\alpha}\,\middle|\,\mathcal{F}_{\tau\wedge\sigma}\right] + e^{-r(\tau\wedge\sigma)}\left(-\gamma_b 1_{\{\tau\le\sigma\}}+\gamma_s 1_{\{\tau\ge\sigma\}}\right)\right)\right]$$
$$= \mathbb{E}^x\left[1_{\{\tau\wedge\sigma<\infty,\,\tau\wedge\sigma<\theta\}}e^{-r(\tau\wedge\sigma)}\left(h(X_{\tau\wedge\sigma})1_{\{\tau<\sigma\}} + g(X_{\tau\wedge\sigma})1_{\{\tau>\sigma\}} + f(X_{\tau\wedge\sigma})1_{\{\tau=\sigma\}}\right)\right]$$
$$= \mathbb{E}^x\left[e^{-r(\tau\wedge\sigma)}\left(h(X_{\tau\wedge\sigma})1_{\{\tau<\sigma\}} + g(X_{\tau\wedge\sigma})1_{\{\tau>\sigma\}} + f(X_{\tau\wedge\sigma})1_{\{\tau=\sigma\}}\right)1_{\{\tau\wedge\sigma<\infty\}}\right] = v(x;\sigma,\tau),$$

where the second to last equality holds because (i) $\tau<\sigma$ or $\tau>\sigma$ implies $\tau\wedge\sigma<\theta$, and (ii) by $f(X_\theta)=0$ we have $f(X_{\tau\wedge\sigma})1_{\{\tau=\sigma,\tau\wedge\sigma<\theta\}} = f(X_{\tau\wedge\sigma})1_{\{\tau=\sigma\}}$ a.s. □

Comparing (2.3) and (2.7), we see that $h(x) = 1_{\{x>0\}}(C(x;-\tilde{p},-\tilde{\alpha})-\gamma_b)$, which means that the buyer receives the CDS value $C(x;-\tilde{p},-\tilde{\alpha})$ at the cost of γ_b if he/she exercises before the seller. For the seller, the payoff of exercising before the buyer is $-g(x) = 1_{\{x>0\}}(C(x;\tilde{p},\tilde{\alpha})-\gamma_s)$. Hence, in both cases the fees γ_b and γ_s can be viewed as strike prices.

Since $C(x;p,\alpha)$ does not depend on (σ,τ), Proposition 2.1 implies that finding the saddle point (σ^*,τ^*) for the Nash equilibrium in (2.2) is equivalent to showing that

$$v(x;\sigma^*,\tau) \le v(x;\sigma^*,\tau^*) \le v(x;\sigma,\tau^*), \quad \forall\,\sigma,\tau\in\mathcal{S}. \tag{2.11}$$

If the Nash equilibrium exists, then the value of the game is $V(x;\sigma^*,\tau^*) = C(x) + v(x;\sigma^*,\tau^*)$, $x\in\mathbb{R}$. According to (2.5), the problem is a step-up (resp. step-down) game when $\tilde{\alpha}<0$ and $\tilde{p}<0$ (resp. $\tilde{\alpha}>0$ and $\tilde{p}>0$).

Remark 2.1. *If $\gamma_b = \gamma_s = 0$, then it follows from (2.7)-(2.9) that $h(x) = g(x) = f(x)$ and*

$$v(x;\sigma,\tau;\tilde{p},\tilde{\alpha},0,0) = \mathbb{E}^x\left[e^{-r(\tau\wedge\sigma)}1_{\{X_{\tau\wedge\sigma}>0,\,\tau\wedge\sigma<\infty\}}C(X_{\tau\wedge\sigma};-\tilde{p},-\tilde{\alpha})\right].$$

In this case, the choice of $\tau^* = \sigma^* = 0$ yields the equilibrium (2.11) with equalities, so the default swap game is always trivially exercised at inception by either party. For similar reasons, we also rule out the trivial case with $\tilde{p} = 0$ or $\tilde{\alpha} = 0$ (even with $\gamma_s + \gamma_b > 0$). Henceforth, we proceed our analysis with $\tilde{p}, \tilde{\alpha} \neq 0$ and $\gamma_b + \gamma_s > 0$.

Next, we point out a symmetry result between the step-up and step-down games.

Proposition 2.2 (symmetry)**.** *For any $\sigma, \tau \in \mathcal{S}$, we have $v(x;\sigma,\tau;\tilde{p},\tilde{\alpha},\gamma_b,\gamma_s) = -v(x;\tau,\sigma;-\tilde{p},-\tilde{\alpha},\gamma_s,\gamma_b)$.*

Proof. First, we deduce from (2.7)-(2.9) that

$$\begin{aligned}
h(x;\tilde{p},\tilde{\alpha},\gamma_b) &= -g(x;-\tilde{p},-\tilde{\alpha},\gamma_b),\\
g(x;\tilde{p},\tilde{\alpha},\gamma_s) &= -h(x;-\tilde{p},-\tilde{\alpha},\gamma_s),\\
f(x;\tilde{p},\tilde{\alpha},\gamma_b,\gamma_s) &= -f(x;-\tilde{p},-\tilde{\alpha},\gamma_s,\gamma_b).
\end{aligned}$$

Substituting these equations to (2.6) of Proposition 2.1, it follows, for every $\tau, \sigma \in \mathcal{S}$, that

$$\begin{aligned}
v(x;\sigma,\tau;\tilde{p},\tilde{\alpha},\gamma_b,\gamma_s) = -\mathbb{E}^x\Big[e^{-r(\tau\wedge\sigma)}\big(&h(X_{\tau\wedge\sigma};-\tilde{p},-\tilde{\alpha},\gamma_s)1_{\{\sigma<\tau\}} + g(X_{\tau\wedge\sigma};-\tilde{p},-\tilde{\alpha},\gamma_b)1_{\{\tau<\sigma\}}\\
&+f(X_{\tau\wedge\sigma};-\tilde{p},-\tilde{\alpha},\gamma_s,\gamma_b)1_{\{\tau=\sigma\}}\big)1_{\{\tau\wedge\sigma<\infty\}}\Big]\\
= -v(x;\tau,\sigma;-\tilde{p},-\tilde{\alpha},\gamma_s,\gamma_b).&
\end{aligned}$$

□

Applying Proposition 2.2 to the Nash equilibrium condition (2.11), we deduce that if (σ^*, τ^*) is the saddle point for the step-down default swap game with $(\tilde{p},\tilde{\alpha},\gamma_b,\gamma_s)$, then the reversed pair (τ^*,σ^*) is the saddle point for the step-up default swap game with $(-\tilde{p},-\tilde{\alpha},\gamma_s,\gamma_b)$. Consequently, it is sufficient to study *either* the step-down *or* the step-up default swap game. This significantly simplifies our analysis.

2.2. **Solution Methods via Continuous and Smooth Fit.** We now present our solution procedure for the optimal stopping game (see (2.11)) via continuous and smooth fit under a general Lévy model. In the next section, we shall focus on the spectrally negative Lévy model and derive an analytical solution. Using the symmetry result from Proposition 2.2, it is sufficient to solve only for the step-down game. Also, we notice from (2.1) that if $\tilde{\alpha} \leq \gamma_s$, then the seller's benefit of a reduced exposure does not exceed the fee, and therefore, should never exercise. As a result, the valuation problem is reduced to a step-down CDS studied in [27], and so we exclude it from our analysis here. With this observation and Remark 2.1, we will proceed with the following assumption without loss of generality:

Assumption 2.1. *We assume that $\tilde{\alpha} > \gamma_s \geq 0$, $\tilde{p} > 0$ and $\gamma_b + \gamma_s > 0$.*

In the step-down game, the protection buyer has an incentive to step-down when default is less likely, or equivalently when X is sufficiently high. On the other hand, the protection seller tends to exercise the step-down option when default is likely to occur, or equivalently when X is sufficiently small. Therefore, we conjecture the following *threshold strategies*, respectively, for the buyer and the seller:

$$\tau_B := \inf\{t \geq 0 : X_t \notin (0, B)\}, \quad \text{and} \quad \sigma_A := \inf\{t \geq 0 : X_t \notin (A, \infty)\},$$

for $B > A \geq 0$. Clearly, $\sigma_A, \tau_B \in \mathcal{S}$. In subsequent sections, we will identify the candidate optimal thresholds A^* and B^*, and verify their optimality rigorously. Meanwhile, for $B > A \geq 0$, we denote

$$\begin{aligned} v_{A,B}(x) &:= v(x; \sigma_A, \tau_B) \\ &= \mathbb{E}^x \left[e^{-r(\tau_B \wedge \sigma_A)} \left(h(X_{\tau_B}) 1_{\{\tau_B < \sigma_A\}} + g(X_{\sigma_A}) 1_{\{\tau_B > \sigma_A\}} + f(X_{\tau_B}) 1_{\{\tau_B = \sigma_A\}} \right) 1_{\{\tau_B \wedge \sigma_A < \infty\}} \right], \end{aligned}$$

for every $x \in \mathbb{R}$. Regarding the last term in the expectation, we note that $\tau_B = \sigma_A$ implies that $\tau_B = \sigma_A = \theta$, and $f(X_\theta) = 0$ a.s., and hence obtain the simplified expression

$$v_{A,B}(x) = \mathbb{E}^x \left[e^{-r(\tau_B \wedge \sigma_A)} \left(h(X_{\tau_B}) 1_{\{\tau_B < \sigma_A\}} + g(X_{\sigma_A}) 1_{\{\tau_B > \sigma_A\}} \right) 1_{\{\tau_B \wedge \sigma_A < \infty\}} \right]. \tag{2.12}$$

For our analysis, it is often more useful to consider the *difference functions*:

$$\Delta_h(x; A, B) := v_{A,B}(x) - h(x) \quad \text{and} \quad \Delta_g(x; A, B) := v_{A,B}(x) - g(x), \quad 0 \leq A < x < B. \tag{2.13}$$

We will identify the candidate exercise thresholds A^* and B^* simultaneously by applying the principle of continuous and smooth fit. Precisely, we determine A^* and B^* from the equations:

$$\text{(continuous fit)} \quad \Delta_h(B-; A, B) = 0 \quad \text{and} \quad \Delta_g(A+; A, B) = 0, \tag{2.14}$$

$$\text{(smooth fit)} \quad \Delta_h'(B-; A, B) = 0 \quad \text{and} \quad \Delta_g'(A+; A, B) = 0, \tag{2.15}$$

where $\Delta_h(B-; A, B) := \lim_{x \uparrow B} \Delta_h(x; A, B)$, $\Delta_h'(B-; A, B) := \lim_{x \uparrow B} \Delta_h'(x; A, B)$, $\Delta_g(A+; A, B) := \lim_{x \downarrow A} \Delta_g(x; A, B)$ and $\Delta_g'(A+; A, B) := \lim_{x \downarrow A} \Delta_g'(x; A, B)$ if these limits exist.

Next, we will verify that σ_{A^*} and τ_{B^*} form a saddle point for the Nash equilibrium (2.11). To this end, we shall prove that

(I) $h(x) \leq v_{A^*,B^*}(x) \leq g(x)$ for every $A^* < x < B^*$;
(II) $e^{-r(t \wedge \sigma_{A^*})} v_{A^*,B^*}(X_{t \wedge \sigma_{A^*}})$ is a supermartingale;
(III) $e^{-r(t \wedge \tau_{B^*})} v_{A^*,B^*}(X_{t \wedge \tau_{B^*}})$ is a submartingale.

After establishing (I)-(III) above, we will apply them to establish (2.11) by showing for the candidate optimal thresholds (A^*, B^*) that

$$v(x; \sigma_{A^*}, \tau) \leq v_{A^*,B^*}(x) \leq v(x; \sigma, \tau_{B^*}), \qquad \forall \sigma, \tau \in \mathcal{S}. \tag{2.16}$$

This will complete the verification that $(\sigma_{A^*}, \tau_{B^*})$ is the saddle point for the Nash equilibrium (see Theorem 3.2 below).

Remark 2.2. *In the last step, it is sufficient to show* (2.16) *holds for all $\tau \in \mathcal{S}_{A^*}$ and $\sigma \in \mathcal{S}_{B^*}$, where*

$$\mathcal{S}_{A^*} := \{\tau \in \mathcal{S} : X_\tau \notin (0, A^*] \ a.s.\} \quad \text{and} \quad \mathcal{S}_{B^*} := \{\sigma \in \mathcal{S} : X_\sigma \notin [B^*, \infty) \ a.s.\}. \tag{2.17}$$

Indeed, for any candidate $\tau \in \mathcal{S}$, we have the domination: $v(x; \sigma_{A^*}, \tau) \leq v(x; \sigma_{A^*}, \hat{\tau})$ for $\hat{\tau} := \tau 1_{\{X_\tau \notin (0,A^*]\}} + \theta 1_{\{X_\tau \in (0,A^*]\}} \in \mathcal{S}_{A^*}$, so the buyer's optimal exercise time τ^* must belong to $\mathcal{S}_{A^*}$. This is intuitive since the seller will end the game as soon as X enters $(0, A^*]$ and hence the buyer should not needlessly stop in this *interval and pay γ_b. Similar arguments apply to the use of $\mathcal{S}_{B^*}$. Then, using the same arguments as for* (2.12)*, we can again safely eliminate the $f(\cdot)$ term in* (2.6) *and write*

$$v(x;\sigma_{A^*},\tau) = \mathbb{E}^x\left[e^{-r(\tau\wedge\sigma_{A^*})}\left(h(X_{\tau\wedge\sigma_{A^*}})1_{\{\tau<\sigma_{A^*}\}} + g(X_{\tau\wedge\sigma_{A^*}})1_{\{\tau>\sigma_{A^*}\}}\right)1_{\{\tau\wedge\sigma_{A^*}<\infty\}}\right], \quad \tau\in\mathcal{S}_{A^*},$$
$$v(x;\sigma,\tau_{B^*}) = \mathbb{E}^x\left[e^{-r(\tau_{B^*}\wedge\sigma)}\left(h(X_{\tau_{B^*}\wedge\sigma)})1_{\{\tau_{B^*}<\sigma\}} + g(X_{\tau_{B^*}\wedge\sigma})1_{\{\tau_{B^*}>\sigma\}}\right)1_{\{\tau_{B^*}\wedge\sigma<\infty\}}\right], \quad \sigma\in\mathcal{S}_{B^*}.$$

3. Solution Methods for the Spectrally Negative Lévy Model

3.1. The Spectrally Negative Lévy Process and the Scale Function.

Let X be a spectrally negative Lévy process with the *Laplace exponent*

$$\phi(s) := \log\mathbb{E}^0\left[e^{sX_1}\right] = cs + \frac{1}{2}\nu^2s^2 + \int_{(0,\infty)}(e^{-sx}-1+sx1_{\{0<x<1\}})\,\Pi(\mathrm{d}x), \quad s\in\mathbb{C}, \tag{3.1}$$

where $c\in\mathbb{R}$, $\nu\geq 0$ is called the Gaussian coefficient, and Π is a Lévy measure on $(0,\infty)$ such that $\int_{(0,\infty)}(1\wedge x^2)\Pi(\mathrm{d}x)<\infty$. See [23], p.212. The risk-neutrality condition requires that $\phi(1)=r$ so that the discounted value is a $(\mathbb{P},\mathbb{F})$-martingale. If the Lévy measure satisfies

$$\int_{(0,\infty)}(1\wedge x)\,\Pi(\mathrm{d}x)<\infty, \tag{3.2}$$

then the Laplace exponent simplifies to

$$\phi(s) = \mu s + \frac{1}{2}\nu^2s^2 + \int_{(0,\infty)}(e^{-sx}-1)\,\Pi(\mathrm{d}x), \quad s\in\mathbb{C}, \tag{3.3}$$

where $\mu := c+\int_{(0,1)}x\,\Pi(\mathrm{d}x)$. Recall that a Lévy process has paths of bounded variation if and only if $\nu=0$ and (3.2) holds. A special example is a compound Poisson process where $\Pi(0,\infty)<\infty$. We ignore the case when X is a negative subordinator (decreasing a.s.). This means that we require μ to be strictly positive if $\nu=0$ and (3.2) holds.

For any spectrally negative Lévy process, there exists a function $W^{(r)}:\mathbb{R}\mapsto\mathbb{R}$, for $r\geq 0$, which is zero on $(-\infty,0)$ and continuous and strictly increasing on $[0,\infty)$. It is characterized by the Laplace transform:

$$\int_0^\infty e^{-sx}W^{(r)}(x)\mathrm{d}x = \frac{1}{\phi(s)-r}, \qquad s>\Phi(r),$$

where Φ is the right inverse of ϕ, defined by

$$\Phi(r) := \sup\{\lambda\geq 0:\phi(\lambda)=r\}, \quad r\geq 0.$$

The function $W^{(r)}$ is often called the *(r-)scale function* in the literature (see e.g. [23]).

Remark 3.1. *There also exists a version of the scale function $W_{\Phi(r)}=\{W_{\Phi(r)}(x);x\in\mathbb{R}\}$ that satisfies*

$$W^{(r)}(x) = e^{\Phi(r)x}W_{\Phi(r)}(x), \quad x\in\mathbb{R}, \tag{3.4}$$

and

$$\int_0^\infty e^{-sx}W_{\Phi(r)}(x)\mathrm{d}x = \frac{1}{\phi(s+\Phi(r))-r}, \quad s>0.$$

The function $W_{\Phi(r)}(x)$ is increasing and

$$W_{\Phi(r)}(x)\uparrow\frac{1}{\phi'(\Phi(r))} \quad \text{as } x\uparrow\infty. \tag{3.5}$$

Using this fact, one can deduce that

$$\frac{W^{(r)\prime}(x)}{W^{(r)}(x)} = \frac{\Phi(r)e^{\Phi(r)x}W_{\Phi(r)}(x) + e^{\Phi(r)x}W'_{\Phi(r)}(x)}{e^{\Phi(r)x}W_{\Phi(r)}(x)} = \frac{\Phi(r)W_{\Phi(r)}(x) + W'_{\Phi(r)}(x)}{W_{\Phi(r)}(x)} \xrightarrow{x\uparrow\infty} \Phi(r). \quad (3.6)$$

From Lemmas 4.3 and 4.4 of [26], we also summarize the behavior of $W^{(r)}$ in the neighborhood of zero.

Lemma 3.1. *For every $r \geq 0$, we have*

$$W^{(r)}(0) = \left\{ \begin{array}{ll} 0, & \textit{unbounded variation} \\ \frac{1}{\mu}, & \textit{bounded variation} \end{array} \right\} \quad \textit{and} \quad W^{(r)\prime}(0+) = \left\{ \begin{array}{ll} \frac{2}{\nu^2}, & \nu > 0 \\ \infty, & \nu = 0 \textit{ and } \Pi(0,\infty) = \infty \\ \frac{r+\Pi(0,\infty)}{\mu^2}, & \textit{compound Poisson} \end{array} \right\}.$$

To facilitate calculations, we define the function

$$Z^{(r)}(x) := 1 + r\int_0^x W^{(r)}(y)\mathrm{d}y, \quad x \in \mathbb{R}$$

which satisfies that

$$\frac{Z^{(r)}(x)}{W^{(r)}(x)} \xrightarrow{x\uparrow\infty} \frac{r}{\Phi(r)}; \quad (3.7)$$

see [23] Exercise 8.5. By Theorem 8.5 of [23], the Laplace transform of θ in (2.4) can be expressed as

$$\zeta(x) = Z^{(r)}(x) - \frac{r}{\Phi(r)}W^{(r)}(x), \quad x \in \mathbb{R}\backslash\{0\}.$$

For the rest of this paper, we assume the following.

Assumption 3.1. *We assume that the Lévy density is completely monotone.*

This is a sufficient condition for the optimality of the threshold strategy (see the proofs of Lemmas 3.6, 3.11-(1) and 3.13-(3) below). In particular, this implies

$$\frac{\partial}{\partial x}\frac{W'_{\Phi(r)}(x)}{W_{\Phi(r)}(x)} \leq 0, \quad x > 0; \quad (3.8)$$

see Remark 3.3 of [27] for its proof. Very similar condition is also used in the related work [28, 29, 30] for the optimal dividend problem under spectrally negative Lévy models. The class of Lévy measures with completely monotone densities is rich. It includes, for example, the variance gamma processes [32, 31], CGMY processes [12], generalized hyperbolic processes [14] and normal inverse Gaussian processes [5]. It also allows for modeling compound-Poisson-type jumps with long-tailed distributions such as the Pareto, Weibull and gamma distributions; see [18]. As shown by [15], its scale function can be approximated arbitrarily closely by those with the hyperexponential Lévy densities; see also Section 4 below.

Let us apply the scale functions to compute the difference functions defined in (2.13). We begin with a lemma.

Lemma 3.2. *For $0 < A < x < B < \infty$, the difference functions in (2.13) are given by*

$$\begin{aligned} \Delta_h(x; A, B) &= \Upsilon(x; A, B) - \left(\frac{\tilde{p}}{r} - \gamma_b\right), \\ \Delta_g(x; A, B) &= \Upsilon(x; A, B) - \left(\frac{\tilde{p}}{r} + \gamma_s\right), \end{aligned} \quad (3.9)$$

where

$$\Upsilon(x;A,B) := \left(\frac{\tilde{p}}{r} - \gamma_b\right)\Lambda_1(x;A,B) + \left(\frac{\tilde{p}}{r} + \gamma_s\right)\Lambda_2(x;A,B) + (\tilde{\alpha} - \gamma_s)\Lambda_3(x;A,B) \tag{3.10}$$

with

$$\Lambda_1(x;A,B) := \mathbb{E}^x\left[e^{-r(\sigma_A\wedge\tau_B)}1_{\{\tau_B<\sigma_A,\,\sigma_A\wedge\tau_B<\infty\}}\right], \tag{3.11}$$

$$\Lambda_2(x;A,B) := \mathbb{E}^x\left[e^{-r(\sigma_A\wedge\tau_B)}1_{\{\tau_B>\sigma_A \text{ or } \sigma_A\wedge\tau_B=\theta\}}1_{\{\sigma_A\wedge\tau_B<\infty\}}\right], \tag{3.12}$$

$$\Lambda_3(x;A,B) := \mathbb{E}^x\left[e^{-r(\sigma_A\wedge\tau_B)}1_{\{\sigma_A\wedge\tau_B=\theta,\,\sigma_A\wedge\tau_B<\infty\}}\right]. \tag{3.13}$$

We observe that $\Delta_h(x;A,B)$ and $\Delta_g(x;A,B)$ are very similar and they possess the common term $\Upsilon(x;A,B)$. This lemma suggests that their determination amounts to computing the expectations Λ_i, $i=1,2,3$ in (3.11)-(3.13).

In the following lemma, we express them in terms of the scale function. Here $\tau_B < \sigma_A$ if and only if it up-crosses B before down-crossing A while $\tau_B > \sigma_A$ or $\sigma_A \wedge \tau_B = \theta$ if and only if it down-crosses A before up-crossing B. Consequently, Λ_1 and Λ_2 can be simply obtained by the scale function (see Theorem 8.1 of [23]). For Λ_3, we require the overshoot distribution that is again obtained via the scale function.

Lemma 3.3. *For $0 < A < x < B < \infty$, the functions Λ_i, $i=1,2,3$ in (3.11)-(3.13) are given by*

$$\Lambda_1(x;A,B) = \frac{W^{(r)}(x-A)}{W^{(r)}(B-A)},$$

$$\Lambda_2(x;A,B) = Z^{(r)}(x-A) - Z^{(r)}(B-A)\frac{W^{(r)}(x-A)}{W^{(r)}(B-A)},$$

$$\Lambda_3(x;A,B) = \frac{W^{(r)}(x-A)}{W^{(r)}(B-A)}\kappa(B;A) - \kappa(x;A),$$

where

$$\begin{aligned}\kappa(x;A) &:= \int_A^\infty \Pi(\mathrm{d}u)\int_0^{u\wedge x-A}\mathrm{d}z W^{(r)}(x-z-A)\\ &= \frac{1}{r}\int_A^\infty \Pi(\mathrm{d}u)\left[Z^{(r)}(x-A) - Z^{(r)}(x-u)\right], \quad x > A > 0.\end{aligned} \tag{3.14}$$

Applying Lemma 3.3, we simplify (3.10) to

$$\Upsilon(x;A,B) = W^{(r)}(x-A)\frac{\Psi(A,B)}{W^{(r)}(B-A)} + \left(\frac{\tilde{p}}{r} + \gamma_s\right)Z^{(r)}(x-A) - (\tilde{\alpha} - \gamma_s)\,\kappa(x;A), \tag{3.15}$$

where

$$\Psi(A,B) := \left(\frac{\tilde{p}}{r} - \gamma_b\right) - \left(\frac{\tilde{p}}{r} + \gamma_s\right)Z^{(r)}(B-A) + (\tilde{\alpha} - \gamma_s)\,\kappa(B;A), \quad 0 < A < B < \infty. \tag{3.16}$$

The function $\Psi(A,B)$ will play a crucial role in the continuous and smooth fit as we discuss in the next subsection.

Remark 3.2. *(1) By taking $B \uparrow \infty$, Lemma 3.2 can be extended to the case when the buyer never exercises and her strategy is θ. By the dominated convergence theorem and that $\tau_B \xrightarrow{B\uparrow\infty} \theta$ a.s., we may write*

$$\Lambda_1(x; A, \infty) := \lim_{B\uparrow\infty} \Lambda_1(x; A, B) = 0,$$
$$\Lambda_2(x; A, \infty) := \lim_{B\uparrow\infty} \Lambda_2(x; A, B) = \mathbb{E}^x\left[e^{-r\sigma_A}\right],$$
$$\Lambda_3(x; A, \infty) := \lim_{B\uparrow\infty} \Lambda_3(x; A, B) = \mathbb{E}^x\left[e^{-r\sigma_A} 1_{\{\sigma_A=\theta,\, \sigma_A<\infty\}}\right].$$

Hence, we can define

$$\Upsilon(x; A, \infty) := \lim_{B\uparrow\infty} \Upsilon(x; A, B) = \left(\frac{\tilde{p}}{r} + \gamma_s\right) \mathbb{E}^x\left[e^{-r\sigma_A}\right] + (\tilde{\alpha} - \gamma_s)\mathbb{E}^x\left[e^{-r\sigma_A} 1_{\{\sigma_A=\theta,\, \sigma_A<\infty\}}\right]. \quad (3.17)$$

Likewise $\Delta_h(x; A, \infty)$ and $\Delta_g(x; A, \infty)$ can be defined in an obvious way.

(2) If we substitute $A = 0$ into (3.10), we obtain

$$\Upsilon(x; 0, B) := \left(\frac{\tilde{p}}{r} - \gamma_b\right) \mathbb{E}^x\left[e^{-r\tau_B} 1_{\{\tau_B<\theta,\, \tau_B<\infty\}}\right] + \left(\frac{\tilde{p}}{r} + \tilde{\alpha}\right) \mathbb{E}^x\left[e^{-r\tau_B} 1_{\{\tau_B=\theta,\, \tau_B<\infty\}}\right], \quad 0 < B \le \infty.$$

As shown in Lemma 3.4 below, $\Upsilon(x; A, B)$ converges to $\Upsilon(x; 0, B)$ as $A \downarrow 0$ if and only if there is not a Gaussian component. Upon the existence of Gaussian component, there is a positive probability of continuously down-crossing (creeping) 0, and the seller tends to exercise immediately before *it reaches 0 rather than not exercising at all.*

Lemma 3.4. *The right-hand limit $\Upsilon(x; 0+, B) := \lim_{A\downarrow 0} \Upsilon(x; A, B)$ is given by*

$$\Upsilon(x; 0+, B) = \Upsilon(x; 0, B) - (\tilde{\alpha} - \gamma_s)\, \mathbb{E}^x\left[e^{-r\tau_B} 1_{\{X_{\tau_B}=0,\, \tau_B<\infty\}}\right], \quad 0 < x < B \le \infty. \quad (3.18)$$

Therefore, $\Upsilon(x; A, B) \xrightarrow{A\downarrow 0} \Upsilon(x; 0, B)$ if and only if the Gaussian coefficient $\nu = 0$.

We also define $\Delta_h(x; 0+, B) := \lim_{A\downarrow 0} \Delta_h(x; A, B)$ and $\Delta_g(x; 0+, B) := \lim_{A\downarrow 0} \Delta_g(x; A, B)$ for all $0 < x < B \le \infty$; see (3.9).

3.2. **Continuous and Smooth Fit.** We shall find the candidate thresholds A^* and B^* by continuous and smooth fit. As we will show and summarize in Table 1 below, the continuous and smooth fit conditions (2.14)-(2.15) will yield the equivalent conditions $\Psi(A^*, B^*) = \psi(A^*, B^*) = 0$ where

$$\begin{aligned}\psi(A, B) &:= \frac{\partial}{\partial B}\Psi(A, B) \\ &= -W^{(r)}(B - A)\,(\tilde{p} + \gamma_s r) + (\tilde{\alpha} - \gamma_s) \int_A^\infty \Pi(\mathrm{d}u) \left(W^{(r)}(B - A) - W^{(r)}(B - u)\right),\end{aligned} \quad (3.19)$$

for all $0 < A < B < \infty$ where (3.19) holds because for every $x > A > 0$

$$Z^{(r)\prime}(x - A) = rW^{(r)}(x - A) \quad \text{and} \quad \kappa'(x; A) = \int_A^\infty \Pi(\mathrm{d}u) \left(W^{(r)}(x - A) - W^{(r)}(x - u)\right). \quad (3.20)$$

Before deriving this result, we need to study the asymptotic behaviors of Ψ and ψ as $B \uparrow \infty$ or $A \downarrow 0$ because there are cases where (1) the buyer never exercises ($B^* = \infty$), (2) the seller never exercises ($A^* = 0$), and (3) the seller delays the exercise until X is arbitrarily close to zero ($A^* = 0+$); see Remark 3.2 and Lemma 3.4. The difference between cases (2) and (3) is explained by Lemma 3.4; upon the existence of Gaussian component, $\Upsilon(x;0,B) > \Upsilon(x;0+,B)$ and the seller exercises at a sufficiently small level $\varepsilon > 0$ (rather than not exercising at all); otherwise, $\Upsilon(x;0,B) = \Upsilon(x;0+,B)$ and the seller may choose θ.

Intuitively speaking, if X jumps downward frequently, then the seller tends to exercise at a level *strictly above* zero. Let us decompose

$$X = X^c + X^d \tag{3.21}$$

where X^c is the continuous martingale (Brownian motion) part and X^d is the jump and drift part of X. Then, the integral

$$\rho(0) := \int_0^\infty \Pi(\mathrm{d}u)\left(1 - e^{-\Phi(r)u}\right)$$

is finite if and only if X^d has paths of bounded variation. As we shall see in Corollary 3.1 below, our candidate threshold level for the seller A^* is always strictly positive if $\rho(0) = \infty$ whether or not there is a Gaussian component. For this reason, we consider the limit as $A \downarrow 0$ only when $\rho(0) < \infty$.

As is clear from (3.16) and (3.19), Ψ and ψ tend to explode as $B \uparrow \infty$. For this reason, we also define the normalized versions:

$$\widehat{\Psi}(A,B) := \frac{\Psi(A,B)}{W^{(r)}(B-A)} \quad \text{and} \quad \widehat{\psi}(A,B) := \frac{\psi(A,B)}{W^{(r)}(B-A)}, \quad 0 < A < B < \infty. \tag{3.22}$$

In order to extend these functions to $A = 0$ and $B = \infty$, we first obtain some asymptotic properties about κ as defined in (3.14). Define

$$\rho(A) := \int_A^\infty \Pi(\mathrm{d}u)\left(1 - e^{-\Phi(r)(u-A)}\right) = \int_0^\infty \Pi(\mathrm{d}u + A)\left(1 - e^{-\Phi(r)u}\right), \quad A > 0.$$

Then $\rho(A)$ is decreasing in A and converges to $\rho(0)$ as $A \downarrow 0$ by the monotone convergence theorem.

Lemma 3.5 (Asymptotics of κ)**.** *(1) For every fixed $x > 0$, $\kappa(x;A)$ is monotonically decreasing in A on $(0,x)$.*

(2) If $\rho(0) < \infty$, then the limit $\kappa(x;0) := \lim_{A\downarrow 0} \kappa(x;A)$ exists and is finite. It can be expressed as

$$\kappa(x;0) = \int_0^\infty \Pi(\mathrm{d}u) \int_0^{u\wedge x} \mathrm{d}z W^{(r)}(x-z) = \frac{1}{r}\int_0^\infty \Pi(\mathrm{d}u)\left[Z^{(r)}(x) - Z^{(r)}(x-u)\right], \quad x > 0. \tag{3.23}$$

(3) For every $A > 0$ (extended to $A \geq 0$ if $\rho(0) < \infty$),

$$\frac{\kappa(x;A)}{W^{(r)}(x-A)} \xrightarrow{x\uparrow\infty} \frac{\rho(A)}{\Phi(r)}.$$

Now, using this lemma, along with (3.16) and (3.19), we extend our definitions of $\widehat{\Psi}(A,B)$ and $\widehat{\psi}(A,B)$ to include the case with $A = 0$ (when $\rho(0) < \infty$) and $B = \infty$, namely, for every $0 \leq A < B \leq \infty$

$$\widehat{\Psi}(A,B) := \begin{cases} \frac{1}{W^{(r)}(B-A)}\left[\left(\frac{\tilde{p}}{r} - \gamma_b\right) - \left(\frac{\tilde{p}}{r} + \gamma_s\right) Z^{(r)}(B-A) + (\tilde{\alpha} - \gamma_s)\,\kappa(B;A)\right], & B < \infty, \\ \frac{1}{\Phi(r)}\left(-(\tilde{p} + r\gamma_s) + (\tilde{\alpha} - \gamma_s)\rho(A)\right), & B = \infty, \end{cases} \tag{3.24}$$

$$\widehat{\psi}(A,B) := \begin{cases} -(\tilde{p} + \gamma_s r) + (\tilde{\alpha} - \gamma_s)\int_A^\infty \Pi(\mathrm{d}u)\left(1 - \frac{W^{(r)}(B-u)}{W^{(r)}(B-A)}\right), & B < \infty, \\ -(\tilde{p} + r\gamma_s) + (\tilde{\alpha} - \gamma_s)\rho(A), & B = \infty. \end{cases} \tag{3.25}$$

Clearly, (3.22) holds true when $0 < A < B < \infty$. We also define $\Psi(0,B)$ and $\psi(0,B)$ for all $0 < B < \infty$ in an obvious way (see Lemma 3.8-(3) below). The finiteness of $\widehat{\Psi}(0,B)$, $\widehat{\Psi}(0,\infty)$ and $\widehat{\psi}(0,\infty)$ when

$\rho(0) < \infty$ is clear by Lemma 3.5-(2). In fact, $\widehat{\psi}(0, B)$ for $0 < B < \infty$ is also finite by the following lemma.

Lemma 3.6. *If $\rho(0) < \infty$, then we have $\int_0^\infty \Pi(\mathrm{d}u)\left(1 - \frac{W^{(r)}(B-u)}{W^{(r)}(B)}\right) < \infty$ for any $0 < B < \infty$.*

The convergence results as $A \downarrow 0$ and $B \uparrow \infty$ are discussed below.

Lemma 3.7 (Asymptotics of Ψ and $\widehat{\Psi}$)**.** *(1) We have $\lim_{B\uparrow\infty} \widehat{\Psi}(A, B) = \widehat{\Psi}(A, \infty)$ for every $A > 0$ (extended to $A \geq 0$ if $\rho(0) < \infty$).*

(2) When $\rho(0) < \infty$, for every $0 < B < \infty$ and $0 < B \leq \infty$, respectively,

$$\Psi(0, B) = \lim_{A\downarrow 0} \Psi(A, B) \quad \textit{and} \quad \widehat{\Psi}(0, B) = \lim_{A\downarrow 0} \widehat{\Psi}(A, B).$$

(3) For every $A > 0$ (extended to $A \geq 0$ if $\rho(0) < \infty$), $\Psi(A, A+) < 0$.

Lemma 3.8 (Properties of $\widehat{\psi}$)**.** *(1) For fixed $0 < B \leq \infty$, $\widehat{\psi}(A, B)$ is decreasing in A on $(0, B)$, and in particular when $\rho(0) < \infty$,*

$$\widehat{\psi}(0, B) = \lim_{A\downarrow 0} \widehat{\psi}(A, B).$$

(2) For fixed $A > 0$ (extended to $A \geq 0$ if $\rho(0) < \infty$), $\widehat{\psi}(A, B)$ is decreasing in B on (A, ∞) and

$$\widehat{\psi}(A, B) \downarrow \widehat{\psi}(A, \infty), \quad \textit{as } B \uparrow \infty. \tag{3.26}$$

(3) The relationship $\psi(0, B) = \partial\Psi(0, B)/\partial B$ also holds for any $0 < B < \infty$ given $\rho(0) < \infty$ where as defined in (3.22)-(3.25)

$$\Psi(0, B) = \left(\frac{\tilde{p}}{r} - \gamma_b\right) - \left(\frac{\tilde{p}}{r} + \gamma_s\right) Z^{(r)}(B) + (\tilde{\alpha} - \gamma_s)\,\kappa(B; 0),$$

$$\psi(0, B) = W^{(r)}(B)\left(-(\tilde{p} + \gamma_s r) + (\tilde{\alpha} - \gamma_s)\int_0^\infty \Pi(\mathrm{d}u)\left(1 - \frac{W^{(r)}(B-u)}{W^{(r)}(B)}\right)\right).$$

Using the above, (3.17) becomes, in view of (3.15) and Lemma 3.7-(1),

$$\Upsilon(x; A, \infty) = W^{(r)}(x - A)\widehat{\Psi}(A, \infty) + \left(\frac{\tilde{p}}{r} + \gamma_s\right) Z^{(r)}(x - A) - (\tilde{\alpha} - \gamma_s)\,\kappa(x; A), \quad 0 < A < x.$$

Similarly, when $\rho(0) < \infty$, (3.18) becomes, in view of (3.15) and Lemmas 3.5-(2) and 3.7-(2),

$$\Upsilon(x; 0+, B) = W^{(r)}(x)\widehat{\Psi}(0, B) + \left(\frac{\tilde{p}}{r} + \gamma_s\right) Z^{(r)}(x) - (\tilde{\alpha} - \gamma_s)\,\kappa(x; 0), \quad 0 < x < B \leq \infty.$$

Figure 1 gives numerical plots of $\Psi(A, \cdot)$, $\widehat{\Psi}(A, \cdot)$, $\psi(A, \cdot)$ and $\widehat{\psi}(A, \cdot)$ for various values of $A > 0$. Lemma 3.8-(1,2) and the fact that $\widehat{\psi}(A, B) \geq 0 \Longleftrightarrow \psi(A, B) \geq 0$ imply that

(a) $\Psi(A, \cdot)$ is monotonically increasing for small A,
(b) $\Psi(A, \cdot)$ is monotonically decreasing for large A,
(c) otherwise $\Psi(A, \cdot)$ is increasing and decreasing.

It can be also confirmed that $\widehat{\Psi}(A, \cdot)$ and $\widehat{\psi}(A, \cdot)$ converge as $B \uparrow \infty$ as in Lemmas 3.7-(1) and 3.8-(2). We shall see that continuous/smooth fit requires (except for the case $A^* = 0$) that $\widehat{\Psi}(A^*, B^*) = \widehat{\psi}(A^*, B^*) = 0$, or equivalently $\Psi(A^*, B^*) = \psi(A^*, B^*) = 0$ when $B^* < \infty$ (attained by the red line in Figure 1).

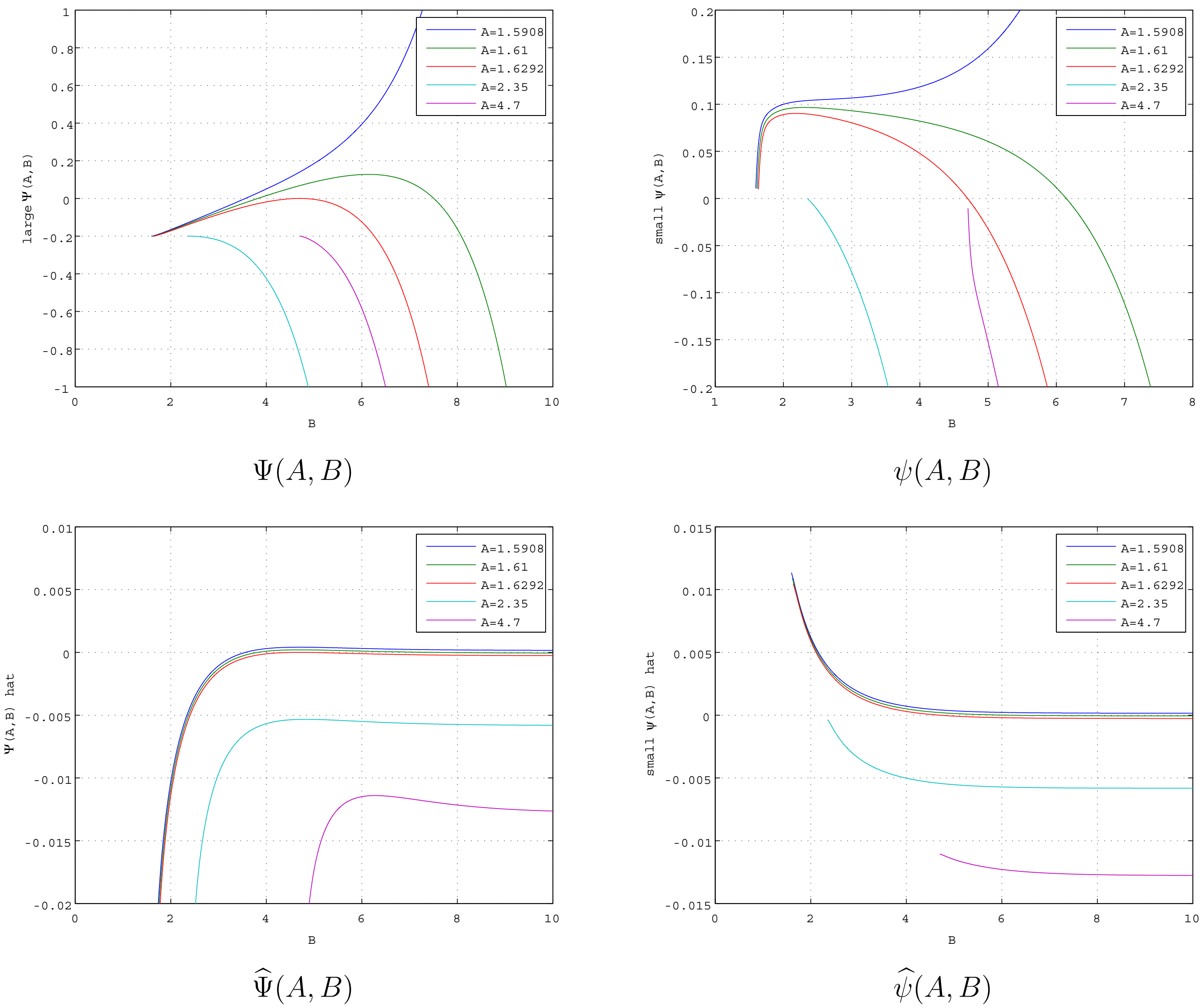


FIGURE 1. Illustration of $\Psi(A,B)$, $\widehat{\Psi}(A,B)$, $\psi(A,B)$, and $\widehat{\psi}(A,B)$ as functions of B.

We are now ready to pursue continuous and smooth fit (2.14)-(2.15). We begin with obtaining the former.
Continuous fit at B: Continuous fit at B is satisfied automatically for all cases since $\Delta_h(B-;A,B)$ exists and

$$\Delta_h(B-;A,B) = \Upsilon(B-;A,B) - \left(\frac{\tilde{p}}{r} - \gamma_b\right) = 0, \quad 0 < A < B < \infty, \tag{3.27}$$

which also holds when $A = 0+$ and $\Delta_h(B-;0+,B) = 0$ given $\rho(0) < \infty$. This is also clear from the fact that any spectrally negative Lévy process creeps upward and hence B is regular for (B,∞) for any arbitrarily level $B > 0$ (see page 212 of [23]).
Continuous fit at A: Similarly, we obtain

$$\Delta_g(A+;A,B) = W^{(r)}(0)\widehat{\Psi}(A,B), \quad 0 < A < B \leq \infty. \tag{3.28}$$

In this case, continuous fit at A holds automatically for the unbounded variation case in view of Lemma 3.1 while it requires

$$\widehat{\Psi}(A,B)=0 \tag{3.29}$$

for the bounded variation case.

We now pursue the smooth fit condition. Substituting (3.20) into the derivative of (3.15), we obtain

$$\Delta_g'(x;A,B)=\Delta_h'(x;A,B)=\Upsilon'(x;A,B)=W^{(r)\prime}(x-A)\widehat{\Psi}(A,B)-\psi(A,x), \tag{3.30}$$

for every $0<A<x<B\leq\infty$.

Smooth fit at B: Applying the smooth fit condition $\Delta_h'(B-;A,B)=0$ to (3.30), smooth fit at $B<\infty$ requires

$$\Gamma(A,B)=0$$

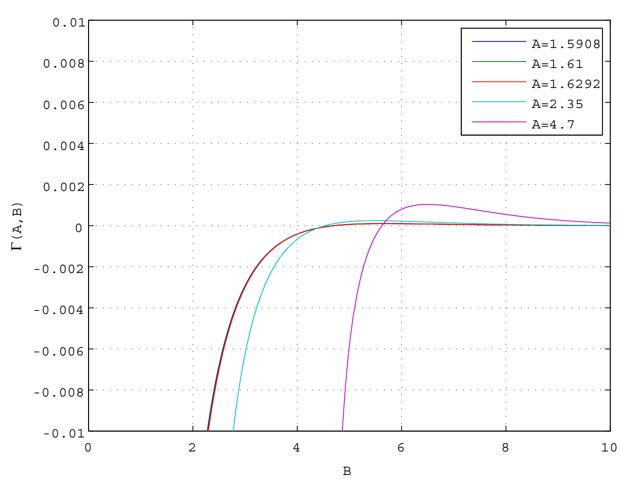
where

$$\Gamma(A,B):=\frac{\Delta_h'(B-;A,B)}{W^{(r)\prime}(B-A)}=\widehat{\Psi}(A,B)-\frac{\psi(A,B)}{W^{(r)\prime}(B-A)},\quad 0\leq A<B<\infty. \tag{3.31}$$

Here we divide $\Delta_h'(B-;A,B)$ by $W^{(r)\prime}(B-A)$ (as in the case $\widehat{\Psi}$ and $\widehat{\psi}$) so that it would not explode as $B\uparrow\infty$. For the case $A=0+$ and $\rho(0)<\infty$, the smooth fit condition $\Delta_h'(B-;0+,B)=0$ requires $\Gamma(0,B)=0$. Figure 2 shows a sample plot of the function $\Gamma(A,\cdot)$. As the following lemma shows, it starts at $-\infty$ (when X is of unbounded variation) and converges to zero as $B\uparrow\infty$.

Lemma 3.9 (Asymptotics of Γ)**.** *The following holds for every $A\geq 0$ (with $\rho(0)<\infty$ when $A=0$).*

(1) There exists $\Gamma(A,\infty):=\lim_{B\uparrow\infty}\Gamma(A,B)=0$.
(2) We have $\Gamma(A,A+):=\lim_{x\downarrow A}\Gamma(A,x)=-\infty$ when X is of unbounded variation.



Γ

FIGURE 2. Illustration of Γ as a function of B.

Smooth fit at A: Assume that it has paths of unbounded variation ($W^{(r)}(0)=0$), then we obtain

$$\Delta_g'(A+;A,B)=W^{(r)\prime}(0+)\widehat{\Psi}(A,B),\quad 0<A<B\leq\infty.$$

	cont. fit at A	smooth fit at A	cont. fit at B	smooth fit at B
bdd. var.	$\widehat{\Psi}(A,B)=0$	N/A	‡	$\Gamma(A,B)=0$
unbdd. var.	‡	$\widehat{\Psi}(A,B)=0$	‡	$\Gamma(A,B)=0$

TABLE 1. Sufficient conditions for continuous and smooth fit. Here ‡ indicates that the condition automatically holds for all (A,B). For the bounded variation case, smooth fit at A is not used and only continuous fit is considered.

Therefore, (3.29) is also a sufficient condition for smooth fit at A for the unbounded variation case.

We summarize the continuous and smooth fit conditions in Table 1. We conclude that

(1) if $\widehat{\Psi}(A,B)=0$, then continuous fit at A holds for the bounded variation case and both continuous and smooth fit at A holds for the unbounded variation case;
(2) if $\Gamma(A,B)=0$, then both continuous and smooth fit conditions at B hold for all cases.

If $\widehat{\Psi}(A,B)=0$ and $\Gamma(A,B)=0$ are simultaneously satisfied, then $\widehat{\psi}(A,B)=0$ is automatically satisfied.

3.3. **Existence and Identification of** (A^*,B^*)**.** In the previous subsection, we have derived the defining equations for the candidate pair (A^*,B^*). Nevertheless, the computation of (A^*,B^*) is non-trivial and depends on the behaviors of functions $\Psi(A,B)$ and $\psi(A,B)$. In this subsection, we prove the existence of (A^*,B^*) and provide a procedure to calculate their values.

Recall from Lemma 3.8-(1) that $\widehat{\psi}(A,\infty)$ is decreasing in A and observe that $\widehat{\psi}(A,A+):=\lim_{x\downarrow A}\widehat{\psi}(A,x)=-(\tilde{p}+r\gamma_s)+(\tilde{\alpha}-\gamma_s)\Pi(A,\infty)$ is also decreasing in A. Hence, let $\underline{A}$ and $\overline{A}$ be the unique values such that

$$\widehat{\psi}(\underline{A},\infty)\equiv-(\tilde{p}+r\gamma_s)+(\tilde{\alpha}-\gamma_s)\rho(\underline{A})=0, \tag{3.32}$$

$$\widehat{\psi}(\overline{A},\overline{A}+)\equiv-(\tilde{p}+r\gamma_s)+(\tilde{\alpha}-\gamma_s)\Pi(\overline{A},\infty)=0, \tag{3.33}$$

upon existence; we set the former zero if $\widehat{\psi}(A,\infty)<0$ for all $A\geq 0$ and also set the latter zero if $\widehat{\psi}(A,A+)<0$ for any $A\geq 0$. Because $\rho(A)\downarrow 0$ and $\Pi(A,\infty)\downarrow 0$ as $A\uparrow\infty$, $\overline{A}$ and $\underline{A}$ are finite. Because $\rho(A)<\Pi(A,\infty)$, we must have $\overline{A}\geq\underline{A}$.

Define for every $\underline{A}\leq A\leq\overline{A}$,

$$\begin{aligned}
\underline{b}(A)&:=\inf\left\{B>A:\widehat{\Psi}(A,B)\geq 0\right\},\\
\overline{b}(A)&:=\inf\left\{B>A:\widehat{\psi}(A,B)\leq 0\right\},\\
b(A)&:=\inf\left\{B>A:\Gamma(A,B)\geq 0\right\},
\end{aligned} \tag{3.34}$$

where we assume $\inf\emptyset=\infty$. Since $W^{(r)}(B-A)>0$ for all $B>A$, we can also write

$$\underline{b}(A)\equiv\inf\{B>A:\Psi(A,B)\geq 0\}\quad\text{and}\quad\overline{b}(A)\equiv\inf\{B>A:\psi(A,B)\leq 0\}.$$

The following theorem shows that there always exists a pair (A^*,B^*) such that one of the following four holds:

case 1: $0<A^*<B^*<\infty$ with $B^*=\underline{b}(A^*)=\overline{b}(A^*)<\infty$;
case 2: $0<A^*<B^*=\infty$ with $B^*=\underline{b}(A^*)=\overline{b}(A^*)=\infty$ and $\widehat{\Psi}(A^*,\infty)=0$;
case 3: $0=A^*<B^*<\infty$ with $B^*=b(0)\leq\underline{b}(0)$;

case 4: $0 = A^* < B^* = \infty$ with $\underline{b}(0) = \infty$ and $b(0) = \infty$.

Theorem 3.1. *(1) If $\underline{A} > 0$ and $\underline{b}(\underline{A}) < \infty$, then there exists $A^* \in (\underline{A}, \overline{A})$ such that $B^* = \underline{b}(A^*) = \overline{b}(A^*) < \infty$. This corresponds to **case 1**.*
*(2) If $\underline{A} > 0$ and $\underline{b}(\underline{A}) = \infty$, then $A^* = \underline{A}$ and $B^* = \infty$ satisfy the condition for **case 2**.*
*(3) If $\underline{A} = 0$, $\overline{A} > 0$, and $\underline{b}(0) < \overline{b}(0)$, then there exists $A^* \in (0, \overline{A})$ such that $B^* = \underline{b}(A^*) = \overline{b}(A^*)$. This corresponds to **case 1**.*
*(4) Suppose (i) $\overline{A} = 0$ or (ii) $\underline{A} = 0$ and $\underline{b}(0) \geq \overline{b}(0)$. If $b(0) < \infty$, then $A^* = 0$ and $B^* = b(0)$ satisfy the condition for **case 3**. If $b(0) = \infty$, then $A^* = 0$ and $B^* = \infty$ satisfy the condition for **case 4**.*

In particular, from (3.32) we infer that $\rho(0) = \infty$ implies $\underline{A} > 0$. This together with Theorem 3.1 obtains the following corollary.

Corollary 3.1. *If X^d as in (3.21) has paths of unbounded variation, then $\rho(0) = \infty$ and $A^* > 0$.*

Remark 3.3. *Note that $\underline{b}(A) = \overline{b}(A)$ implies $b(A) = \underline{b}(A) = \overline{b}(A)$ in view of (3.31) (even when they are $+\infty$; see Lemma 3.9-(1)). By construction (see (3.34)), A^* and B^* obtained above satisfy the following.*
(1) For every $A^* < B < B^*$,

$$\widehat{\Psi}(A^*, B) < 0 \quad \textit{and} \quad \Gamma(A^*, B) < 0. \tag{3.35}$$

(2) If $A^* > 0$, $\widehat{\Psi}(A^*, B^*) = 0$ (i.e., continuous or smooth fit at A^* is satisfied).
(3) $\Gamma(A^*, B^*) = 0$ (i.e., continuous and smooth fit at B^* is satisfied).

In Theorem 3.1, in case of (1) and (3), we further need to identify A^* and B^*. Here we use the following properties.

Lemma 3.10. *(1) $\underline{b}(A)$ increases in A on $(\underline{A}, \overline{A})$,*
(2) $\overline{b}(A)$ decreases in A on $(\underline{A}, \overline{A})$.

This lemma implies that (i) if $\overline{b}(A) > \underline{b}(A)$, then A^* must lie on $(A, \overline{A})$ and (ii) if $\overline{b}(A) < \underline{b}(A)$, then A^* must lie on $(\underline{A}, A)$. By Lemma 3.10 and Theorem 3.1, the following algorithm, motivated by the bisection method, is guaranteed to output the pair (A^*, B^*). Here let $\varepsilon > 0$ be the error parameter.

Step 1: Compute $\underline{A}$ and $\overline{A}$.
- **Step 1-1:** If (i) $\overline{A} = 0$ or (ii) $\underline{A} = 0$ and $\underline{b}(0) \geq \overline{b}(0)$, then stop and conclude that this is **case 3** or **4** with $A^* = 0$ and $B^* = b(0)$.
- **Step 1-2:** If $\underline{A} > 0$ and $\underline{b}(\underline{A}) = \infty$, then stop and conclude that this is **case 2** with $A^* = \underline{A}$ and $B^* = \infty$.

Step 2: Set $A = (\underline{A} + \overline{A})/2$.
Step 3: Compute $\overline{b}(A)$ and $\underline{b}(A)$.
- **Step 3-1:** If $|\overline{b}(A) - \underline{b}(A)| \leq \varepsilon$, then stop and conclude that this is **case 1** with $A^* = A$ and $B^* = \underline{b}(A)$ (or $B^* = \overline{b}(A)$).
- **Step 3-2:** If $|\overline{b}(A) - \underline{b}(A)| > \varepsilon$ and $\overline{b}(A) > \underline{b}(A)$, then set $\underline{A} = A$ and go back to **Step 2**.
- **Step 3-3:** If $|\overline{b}(A) - \underline{b}(A)| > \varepsilon$ and $\overline{b}(A) < \underline{b}(A)$, then set $\overline{A} = A$ and go back to **Step 2**.

3.4. **Verification of Equilibrium.** Our candidate value function for the Nash equilibrium is given by (2.12) with A^* and B^* obtained by the procedure above. Suppose $A^* > 0$. Then $(\sigma_{A^*}, \tau_{B^*})$ is the candidate saddle point and the corresponding expected value is given by

$$v_{A^*,B^*}(x) = \left\{ \begin{array}{ll} h(x), & x \geq B^* \\ h(x) + \Delta_h(x; A^*, B^*), & A^* < x < B^* \\ g(x), & x \leq A^* \end{array} \right\} = -\left(\frac{\tilde{p}}{r} + \tilde{\alpha}\right)\zeta(x) + J(x) \tag{3.36}$$

where

$$J(x) := \begin{cases} \frac{\tilde{p}}{r} - \gamma_b, & x \geq B^*, \\ \Upsilon(x; A^*, B^*), & A^* < x < B^*, \\ \frac{\tilde{p}}{r} + \gamma_s, & 0 < x \leq A^*, \\ \frac{\tilde{p}}{r} + \tilde{\alpha} & x \leq 0. \end{cases}$$

Here when $B^* = \infty$, the buyer's strategy is θ.

Suppose $A^* = 0$. By Corollary 3.1, this excludes the case when X^d (the jump and drift part of X) is of unbounded variation. If $\nu = 0$, then we will show that the candidate pair (θ, τ_{B^*}) forms a saddle point for the Nash equilibrium (2.11), and the expected value v_{0,B^*} is given by (3.36) with A^* replaced with 0. In contrast, if $\nu > 0$, then Nash equilibrium (2.11) does not exist, but an alternative form of "equilibrium" is attained. Recall Remark 3.2 and Lemma 3.4. Specifically, we have

$$v(x; \sigma_{0+}, \tau) \leq v_{0+,B^*}(x) \leq v(x; \sigma, \tau_{B^*}), \quad \sigma, \tau \in \mathcal{S}, \tag{3.37}$$

where

$$v(x; \sigma_{0+}, \tau) := \mathbb{E}^x\left[e^{-r\tau}\left(h(X_\tau) - (\tilde{\alpha} - \gamma_s)1_{\{X_\tau = 0\}}\right)1_{\{\tau < \infty\}}\right], \quad \tau \in \mathcal{S},$$
$$v_{0+,B^*}(x) := \mathbb{E}^x\left[e^{-r\tau_{B^*}}\left(h(X_{\tau_{B^*}}) - (\tilde{\alpha} - \gamma_s)1_{\{X_{\tau_{B^*}} = 0\}}\right)1_{\{\tau_{B^*} < \infty\}}\right].$$

Here, the value functions $v(x; \sigma_{0+}, \tau)$ and $v_{0+,B^*}(x)$ correspond to the scenario where the seller exercises *immediately before* θ when X continuously down-crosses the level zero. However, since this exercise timing cannot be represented by any stopping time, even though it can be approximated arbitrarily closely by σ_ε for $\varepsilon > 0$ sufficiently small, (3.37) is not a Nash equilibrium.

Theorem 3.2. *(1) In **cases 1** and **2** ($A^* > 0$), Nash equilibrium exists with saddle point $(\sigma_{A^*}, \tau_{B^*})$ and its expected value v_{A^*,B^*} given by (3.36). In other words,*

$$v(x; \sigma_{A^*}, \tau) \leq v_{A^*,B^*}(x) \leq v(x; \sigma, \tau_{B^*}), \quad \forall \sigma, \tau \in \mathcal{S}. \tag{3.38}$$

*(2) In **cases 3** and **4** ($A^* = 0$),*

- *(a) if $\nu = 0$, Nash equilibrium exists with saddle point (θ, τ_{B^*}) and its expected value v_{A^*,B^*} given by (3.36), and (3.38) holds;*
- *(b) if $\nu > 0$, then the alternative equilibrium (3.37) holds, and the value function satisfies $v_{0+,B^*}(x) = \lim_{\varepsilon \downarrow 0} v(x; \sigma_\varepsilon, \tau_{B^*})$.*

With this theorem, the value of the step-down game is recovered by $V(x) = C(x) + v(x)$ by Proposition 2.1 and that of the step-up game is recovered by $V(x) = C(x) - v(x)$ by Proposition 2.2.

The proof of the theorem involves the crucial steps:

(i) domination property

(a) $\mathbb{E}^x\left[e^{-r(\tau\wedge\sigma_{A^*})}v_{A^*,B^*}(X_{\tau\wedge\sigma_{A^*}})1_{\{\tau\wedge\sigma_{A^*}<\infty\}}\right]\geq v(x;\sigma_{A^*},\tau)$ for all $\tau\in\mathcal{S}$;

(b) $\mathbb{E}^x\left[e^{-r(\sigma\wedge\tau_{B^*})}v_{A^*,B^*}(X_{\sigma\wedge\tau_{B^*}})1_{\{\sigma\wedge\tau_{B^*}<\infty\}}\right]\leq v(x;\sigma,\tau_{B^*})$ for all $\sigma\in\mathcal{S}$;

(ii) sub/super-harmonic property

(a) $(\mathcal{L}-r)v_{A^*,B^*}(x)>0$ for every $0<x<A^*$;

(b) $(\mathcal{L}-r)v_{A^*,B^*}(x)=0$ for every $A^*<x<B^*$;

(c) $(\mathcal{L}-r)v_{A^*,B^*}(x)<0$ for every $x>B^*$.

Here $\mathcal{L}$ is the infinitesimal generator associated with the process X

$$\mathcal{L}f(x)=cf'(x)+\frac{1}{2}\sigma^2f''(x)+\int_0^\infty\left[f(x-z)-f(x)+f'(x)z1_{\{0<z<1\}}\right]\Pi(\mathrm{d}z)$$

applied to sufficiently smooth function f that is C^2 when $\nu>0$ and C^1 otherwise.

We prove them in the following lemmas.

Lemma 3.11. *For every $x\in(A^*,B^*)$, the following inequalities hold:*

$$\Delta_g(x;A^*,B^*)\leq 0, \tag{3.39}$$

$$\Delta_h(x;A^*,B^*)\geq 0, \tag{3.40}$$

where it is understood for the case $A^*=0$ and $\nu>0$ that the above results hold with $A^*=0+$.

Applying this lemma and the definitions of $\mathcal{S}_{A^*}$ and $\mathcal{S}_{B^*}$ in (2.17) of Remark 2.2, we have the following.

Lemma 3.12. *Fix $x>0$.*

(1) For every $\tau\in\mathcal{S}_{A^*}$, when $A^*>0$

$$g(X_{\sigma_{A^*}\wedge\tau})1_{\{\sigma_{A^*}<\tau\}}+h(X_{\sigma_{A^*}\wedge\tau})1_{\{\tau<\sigma_{A^*}\}}\leq v_{A^*,B^*}(X_{\sigma_{A^*}\wedge\tau}),\quad \mathbb{P}^x-a.s.\ on\ \{\sigma_{A^*}\wedge\tau<\infty\},$$

and when $A^*=0$,

$$-(\tilde{\alpha}-\gamma_s)1_{\{X_\tau=0\}}+h(X_\tau)1_{\{\tau<\theta\}}\leq v_{0+,B^*}(X_\tau),\quad \mathbb{P}^x-a.s.\ \ on\ \{\tau<\infty\}.$$

(2) For every $\sigma\in\mathcal{S}_{B^*}$,

$$g(X_{\sigma\wedge\tau_{B^*}})1_{\{\sigma<\tau_{B^*}\}}+h(X_{\sigma\wedge\tau_{B^*}})1_{\{\tau_{B^*}<\sigma\}}\geq v_{A^*,B^*}(X_{\sigma\wedge\tau_{B^*}}),\quad \mathbb{P}^x-a.s.\ \ on\ \{\sigma\wedge\tau_{B^*}<\infty\},$$

where it is understood for the case $A^*=0$ and $\nu>0$ that the above holds with $A^*=0+$.

Lemma 3.13. *(1) When $A^*>0$, we have $(\mathcal{L}-r)v_{A^*,B^*}(x)>0$ for every $0<x<A^*$.*

(2) We have $(\mathcal{L}-r)v_{A^*,B^*}(x)=0$ for every $A^*<x<B^*$.

(3) When $B^*<\infty$, we have $(\mathcal{L}-r)v_{A^*,B^*}(x)<0$ for every $x>B^*$.

With the help of Lemmas 3.12 and 3.13 above, we provide the rest of the proof of Theorem 3.2 in the Appendix.

4. Hyperexponential Jumps and Numerical Examples

In this section, we consider spectrally negative Lévy processes with i.i.d. hyperexponential jumps and provide some numerical examples to illustrate the buyer's and seller's optimal exercise strategies and the fair premium behaviors. Since the Lévy density is assumed to be completely monotone, it can be approximated arbitrarily closely by hyperexponential densities (see, e.g., [15, 18]). In a related work, Asmussen et al. [2] approximate the Lévy density of the CGMY process by a hyperexponential density. Herein, we will use the explicit expression of the scale function obtained by [15].

4.1. **Spectrally Negative Lévy Processes with Hyperexponential Jumps.** Let X be a spectrally negative Lévy process of the form

$$X_t - X_0 = \mu t + \nu B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \le t < \infty.$$

Here $B = \{B_t; t \ge 0\}$ is a standard Brownian motion, $N = \{N_t; t \ge 0\}$ is a Poisson process with arrival rate λ, and $Z = \{Z_n; n = 1, 2, \ldots\}$ is an i.i.d. sequence of hyperexponential random variables with density function

$$f(z) := \sum_{i=1}^{m} \alpha_i \eta_i e^{-\eta_i z}, \quad z > 0,$$

for some $0 < \eta_1 < \cdots < \eta_m < \infty$. Clearly, the corresponding Lévy density λf is completely monotone. Its Laplace exponent (3.1) is given by

$$\phi(s) = \mu s + \frac{1}{2}\nu^2 s^2 - \lambda \sum_{i=1}^{m} \alpha_i \frac{s}{\eta_i + s}.$$

For our examples, we assume $\nu > 0$. In this case, there are $m+1$ *negative* solutions to the equation $\phi(s) = r$ and their absolute values $\{\xi_{i,r}; i = 1, \ldots, m+1\}$ satisfy the interlacing condition:

$$0 < \xi_{1,r} < \eta_1 < \xi_{2,r} < \cdots < \eta_m < \xi_{m+1,r} < \infty.$$

For this process, the scale functions and its derivative are given by for every $x \ge 0$

$$\begin{aligned} W^{(r)}(x) &= \sum_{i=1}^{m+1} C_i \left[e^{\Phi(r)x} - e^{-\xi_{i,r}x} \right], \\ W^{(r)\prime}(x) &= \sum_{i=1}^{m+1} C_i \left[\Phi(x) e^{\Phi(r)x} + \xi_{i,r} e^{-\xi_{i,r}x} \right], \\ Z^{(r)}(x) &= 1 + r \sum_{i=1}^{m+1} C_i \left[\frac{1}{\Phi(r)} \left(e^{\Phi(r)x} - 1 \right) + \frac{1}{\xi_{i,r}} \left(e^{-\xi_{i,r}x} - 1 \right) \right] \end{aligned} \tag{4.1}$$

where

$$C_i := A_{i,r} \frac{2}{\nu^2 \sum_{i=1}^{m+1} A_{i,r} \xi_{i,r}} \left(\frac{\xi_{i,r}}{\Phi(r) + \xi_{i,r}} \right), \quad 1 \le i \le m+1,$$

$$A_{k,r} := \frac{\prod_{j \in \{1,\ldots,m\}} \left(1 - \frac{\xi_{k,r}}{\eta_j} \right)}{\prod_{i \in \{1,\ldots,m\} \setminus \{k\}} \left(1 - \frac{\xi_{k,r}}{\xi_{i,r}} \right)}, \quad 1 \le k \le m+1.$$

In addition, applying (4.1) to (3.4) yields

$$W_{\Phi(r)}(x) = \sum_{i=1}^{m+1} C_i \left[1 - e^{-(\Phi(r)+\xi_{i,r})x}\right],$$

which is concave in x, with the limit $W_{\Phi(r)}(\infty) = \sum_{i=1}^{m+1} C_i$, which equals $(\phi'(\Phi(r)))^{-1}$ by (3.5).

Recall that, in contrast to $\psi(A,B)$ and $\Psi(A,B)$, $\widehat{\psi}(A,B)$ and $\widehat{\Psi}(A,B)$ do not explode. Therefore, they are used to compute the optimal thresholds A^* and B^* and the value function V. Below we provide the formulas for $\widehat{\psi}(A,B)$ and $\widehat{\Psi}(A,B)$. The computations are very tedious but straightforward, so we omit the proofs here.

In summary, for $B > A \geq 0$, we have

$$\widehat{\psi}(A,B) = -(\tilde{p} + \gamma_s r) + (\tilde{\alpha} - \gamma_s)\lambda \sum_{j=1}^{m} \alpha_j e^{-\eta_j A} - \frac{W_{\Phi(r)}(\infty)}{W_{\Phi(r)}(B-A)}(\tilde{\alpha} - \gamma_s)\lambda \sum_{j=1}^{m} \alpha_j \frac{\eta_j}{\Phi(r)+\eta_j} e^{-\eta_j A}$$
$$+ \frac{\tilde{\alpha} - \gamma_s}{W_{\Phi(r)}(B-A)} \lambda e^{-\Phi(r)(B-A)} \sum_{i=1}^{m+1} \sum_{j=1}^{m} \alpha_j C_i \left[\frac{\eta_j}{\Phi(r)+\eta_j} e^{-\eta_j B} + \frac{\eta_j}{\xi_{i,r} - \eta_j}\left(e^{-\eta_j B} - e^{-\xi_{i,r}(B-A)-\eta_j A}\right)\right]$$

and

$$\widehat{\Psi}(A,B) = \frac{1}{W_{\Phi(r)}(B-A)} \times$$
$$\left[-(\tilde{p} + \gamma_s r)\frac{1}{\Phi(r)} W_{\Phi(r)}(\infty) + \lambda W_{\Phi(r)}(\infty)(\tilde{\alpha} - \gamma_s) \sum_{j=1}^{m} \alpha_j e^{-\eta_j A} \frac{1}{\Phi(r)+\eta_j} + e^{-\Phi(r)(B-A)} \varrho(A,B)\right]$$

where

$$\varrho(A,B) := (\tilde{\alpha} - \gamma_s)\lambda \sum_{j=1}^{m} \alpha_j \sum_{i=1}^{m+1} C_i \left(e^{-\eta_j B}\left[-\frac{1}{\Phi(r)+\eta_j} + \frac{1}{\xi_{i,r} - \eta_j}\right] - e^{-\eta_j A - \xi_{i,r}(B-A)} \frac{1}{\xi_{i,r} - \eta_j}\right)$$
$$- (\tilde{p} + \gamma_s r) \sum_{i=1}^{m+1} C_i \left[-\frac{1}{\Phi(r)} + \frac{1}{\xi_{i,r}}\left(e^{-\xi_{i,r}(B-A)} - 1\right)\right] - (\gamma_b + \gamma_s).$$

Also, setting $B = \infty$ and $B = A+$, (3.32)-(3.33) yields

$$\widehat{\psi}(A,\infty) = -(\tilde{p} + \gamma_s r) + \Phi(r)\lambda(\tilde{\alpha} - \gamma_s) \sum_{j=1}^{m} \frac{\alpha_j}{\Phi(r)+\eta_j} e^{-\eta_j A},$$
$$\widehat{\psi}(A,A+) = -(\tilde{p} + \gamma_s r) + (\tilde{\alpha} - \gamma_s)\lambda \sum_{j=1}^{m} \alpha_j e^{-\eta_j A}.$$

4.2. **Numerical Results.** In our numerical example, we focus on the exponential jump case with $m = 1$ so as to conduct sensitivity analysis. We assume that f is an exponential density with parameter $\eta > 0$. This simple density specification allows for more intuitive interpretation of our numerical results. Let us denote the step-up/down ratio by $q := \hat{p}/p = \hat{\alpha}/\alpha$. We consider four contract specifications:

(C) cancellation game with $q = 0$ (position canceled at exercise),
(D) step-down game with $q = 0.5$ (position halved at exercise),
(V) vanilla CDS with $q = 1.0$ (position unchanged at exercise),

(U) step-up game with $q = 1.5$ (position raised at exercise).

The model parameters are $r = 0.03$, $\lambda = 1.0$, $\eta = 2.0$, $\nu = 0.2$, $\alpha = 1$, $x = 1.5$ and $\gamma_s = \gamma_b = 1000$ bps, unless specified otherwise. We also choose μ so that the risk-neutral condition $\phi(1) = r$ is satisfied.

Figure 3 shows for all four cases the contract value V to the buyer as a function of x given a fixed premium rate. It is decreasing in x since default is less likely for higher value of x. For the cancellation game, V takes the constant values $\gamma_s = 1000$ bps for $x \leq A^*$ and $-\gamma_b = -1000$ bps for $x \geq B^*$ since in these regions immediate cancellation with a fee is optimal.

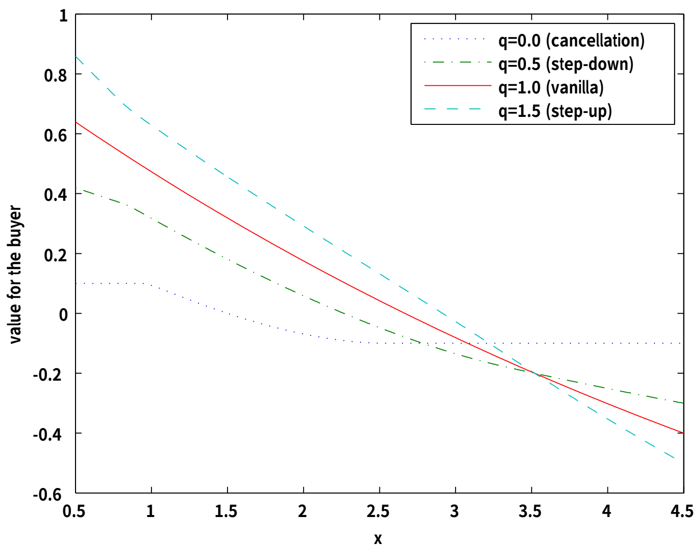
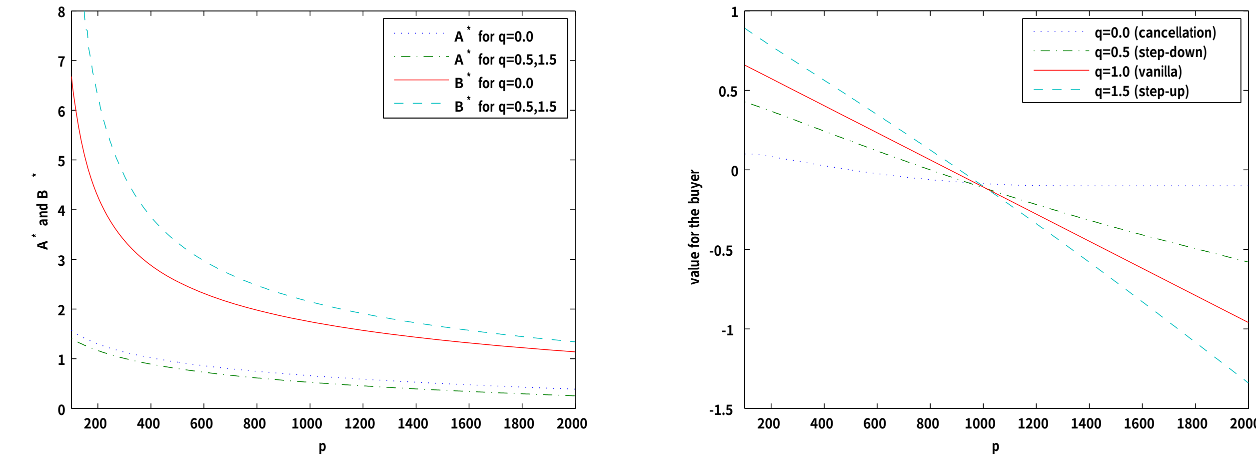


FIGURE 3. The value for the buyer $V(x; \sigma_{A^*}, \tau_{B^*})$ as a function of x. Here $r = 0.03$, $p = 0.05$, $\mu = 0.1352$, $\lambda = 1.0$, $\eta = 2.0$, $\nu = 0.2$, and $\gamma_b = \gamma_s = 1000$ bps.

In Figure 4, we show the optimal thresholds A^* and B^* and the value V with respect to p. The symmetry argument discussed in Section 2 applies to the cases (D) and (U). As a result, the A^* in (D) is identical to the B^* in (U), and the B^* in (D) is identical to the A^* in (U). In all four cases, both A^* and B^* are decreasing in p. In other words, as p increases, the buyer tends to exercise earlier while the seller tends to delay exercise. Intuitively, a higher premium makes waiting more costly for the buyer but more profitable for the seller. The value V in the cancellation game stays constants when p is sufficiently small because the seller would exercise immediately; it also becomes flat when p is sufficiently high because the buyer would exercise immediately.

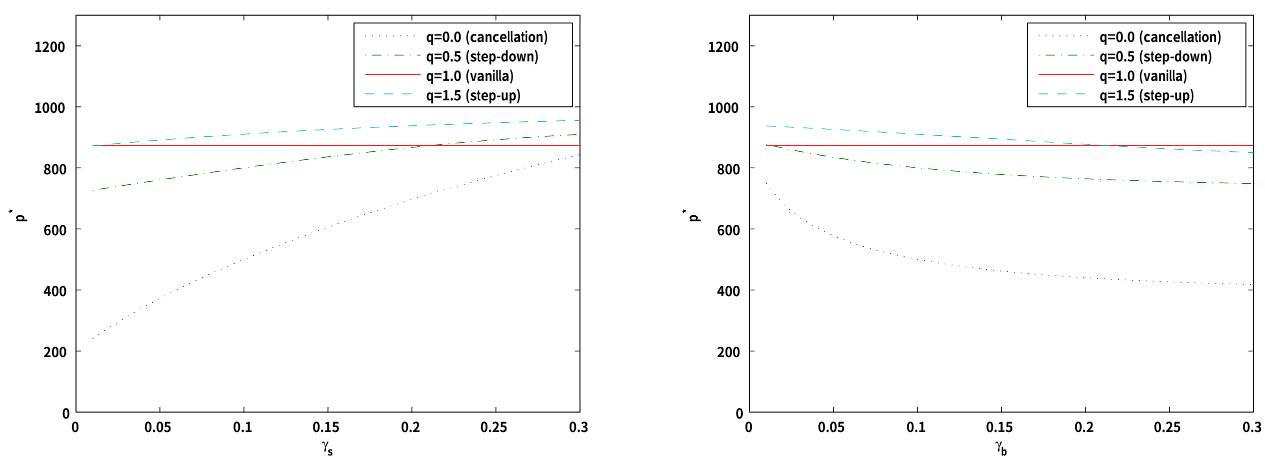
As illustrated in Figure 4-(b), the value V (from the buyer's perspective) is always decreasing in p. Using a bisection method, we numerically determine the fair premium p^* so that $V = 0$. We illustrate in Figure 5 the fair premium p^* as a function of γ_b and γ_s. As is intuitive, the fair premium p^* is increasing in γ_s and decreasing in γ_b. Figures 6 and 7 show the fair premium p^* with respect to x for various values of λ and η, respectively. In all cases, a higher x implies a lower p^* due to a lower default likelihood. It is also increasing in λ and decreasing in η for similar reasons. Figure 8 shows the same plots for various values of ν. As can be observed, p^* is increasing in ν. This is a result of the risk-neutral condition and how μ is chosen: μ decreases as ν increases and the overall drift of the process X decreases and so default comes



(a) A^* and B^* w.r.t. p (b) V w.r.t. p

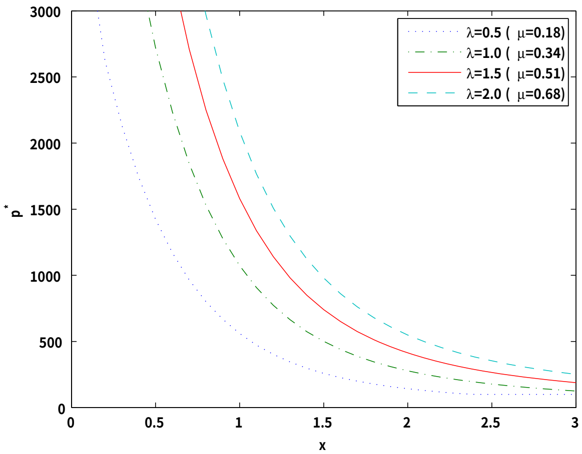
FIGURE 4. Optimal threshold levels A^* and B^* and the value for the buyer with respect to p. The parameters are $r = 0.03$, $x = 1.5$, $\mu = 0.3433$, $\lambda = 0.5$, $\eta = 2.0$, $\nu = 0.2$, and $\gamma_b = \gamma_s = 1000$ bps.

more likely. It should be noted that ν measures the fluctuation of the continuous part of X. When μ is fixed, the increment of ν means less surprise in the evolution of X, which is favorable to the seller.

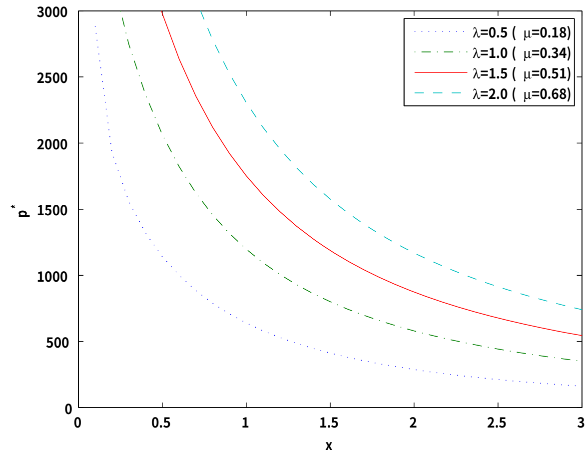


(a) p^* w.r.t. γ_s (b) p^* w.r.t. γ_b

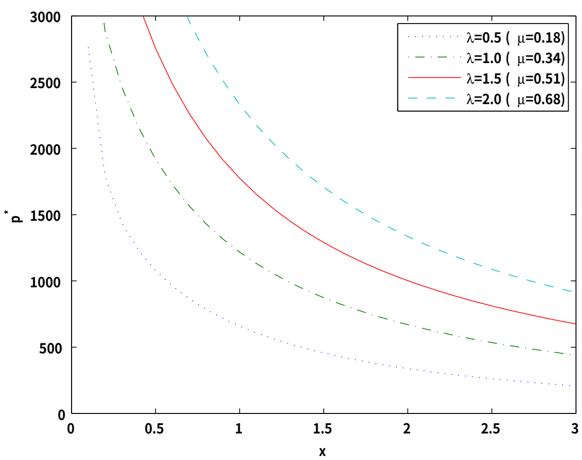
FIGURE 5. Fair premium with respect to γ_b and γ_s. Here $r = 0.03$, $x = 1.5$, $\mu = 0.3433$, $\lambda = 1.0$, $\eta = 2.0$, $\nu = 0.2$, and $\gamma_b = \gamma_s = 1000$ bps if not specified otherwise.



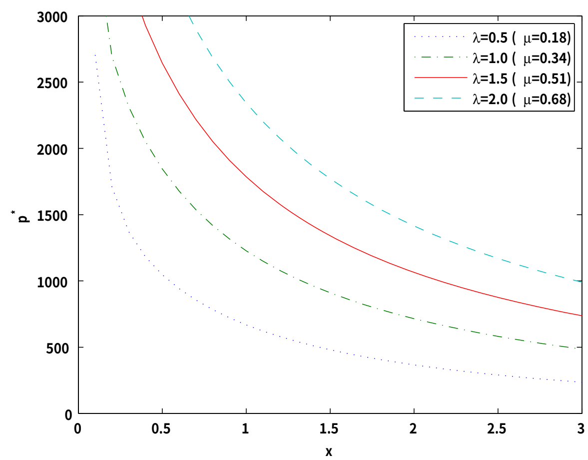
(a) $q = 0$ (cancellation)



(b) $q = 0.5$ (step-down)

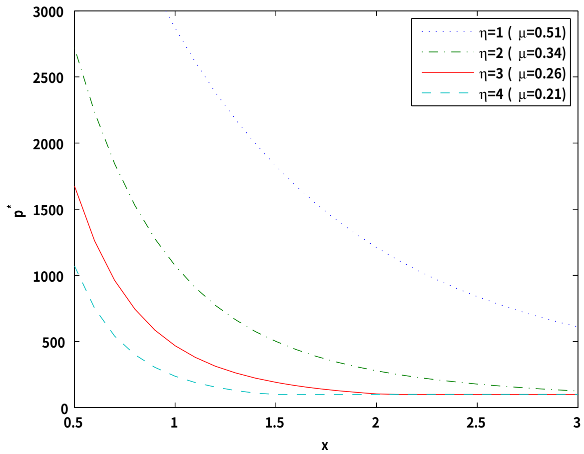


(c) $q = 1.0$ (vanilla)

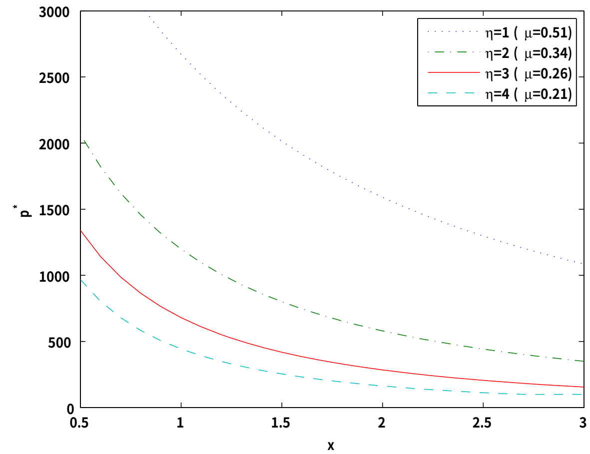


(d) $q = 1.5$ (step-up)

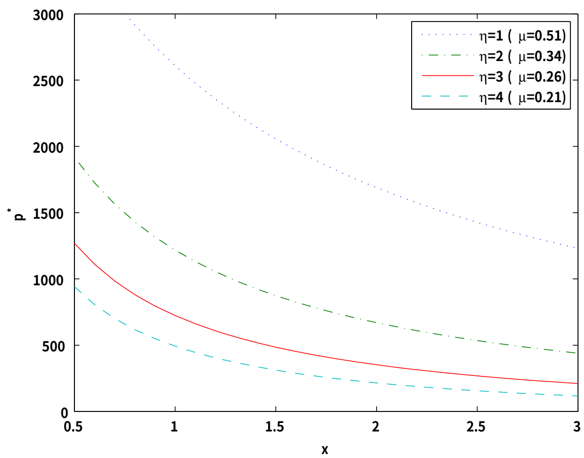
FIGURE 6. Fair premium with respect to x with various values of the jump intensity λ. Here $r = 0.03$, $\eta = 2.0$, $\nu = 0.2$, and $\gamma_b = \gamma_s = 1000$ bps.



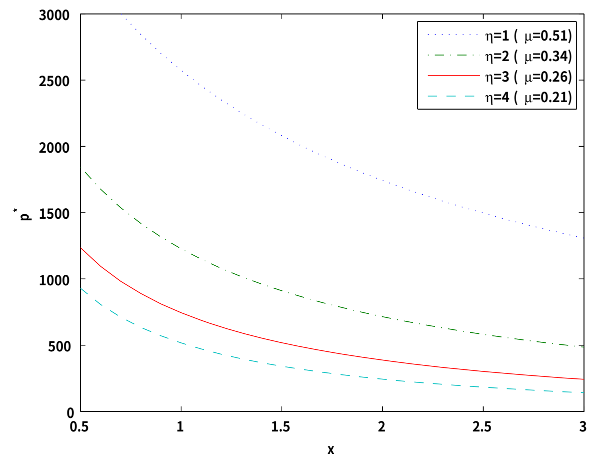
(a) $q = 0$ (cancellation)



(b) $q = 0.5$ (step-down)

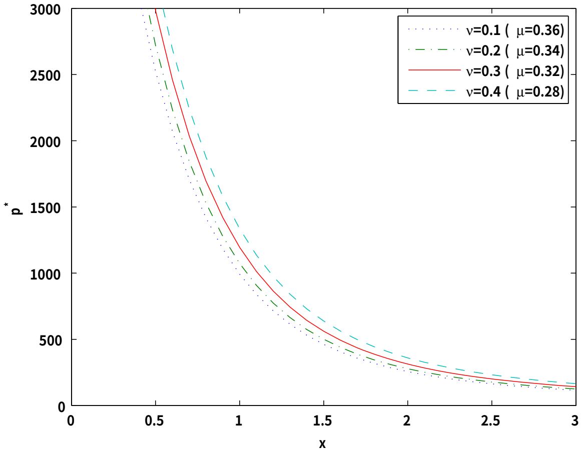


(c) $q = 1.0$ (vanilla)

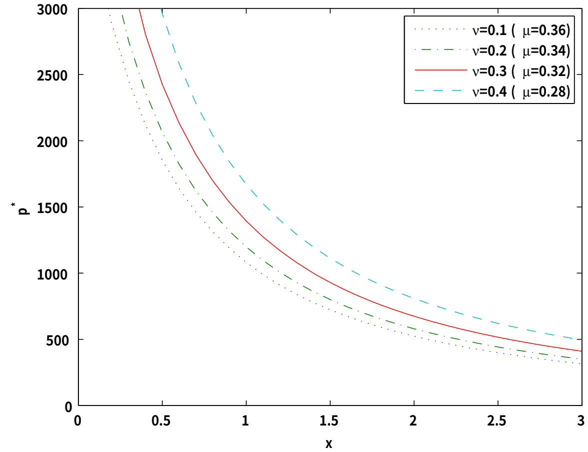


(d) $q = 1.5$ (step-up)

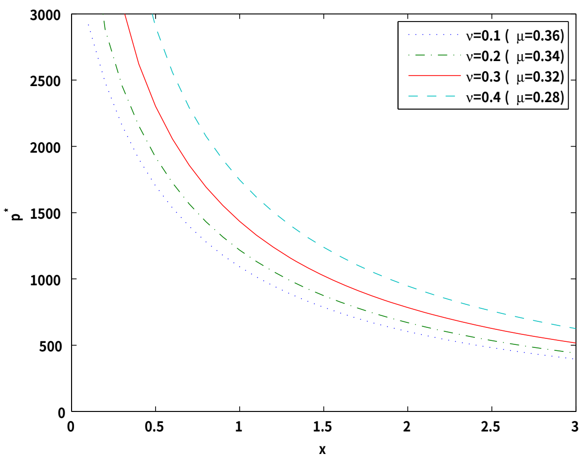
FIGURE 7. Fair premium with respect to x with various values of the jump parameter η. Here $r = 0.03$, $\lambda = 1.0$, $\nu = 0.2$, and $\gamma_b = \gamma_s = 1000$ bps.



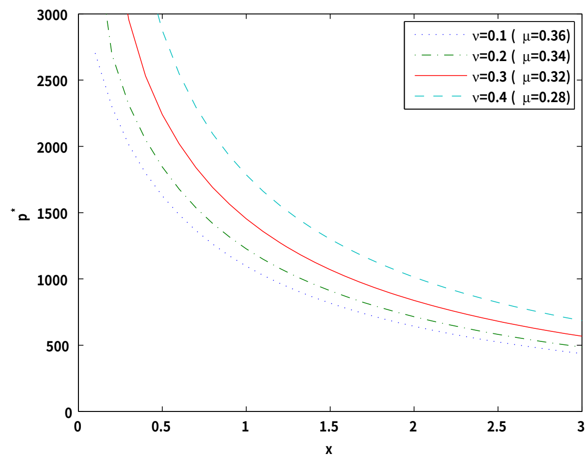
(a) $q = 0$ (cancellation)



(b) $q = 0.5$ (step-down)



(c) $q = 1.0$ (vanilla)



(d) $q = 1.5$ (step-up)

FIGURE 8. Fair premium with respect to x with various values of the jump parameter ν. Here $r = 0.03$, $\lambda = 1.0$, $\eta = 2$, and $\gamma_b = \gamma_s = 1000$ bps.

5. Conclusions

We have discussed the valuation of a default swap contract where the protection buyer and seller can alter the respective position once prior to default. This new contractual feature drives the protection buyer/seller to consider the optimal timing to control credit risk exposure. The valuation problem involves the analytical and numerical studies of an optimal stopping game with early termination from default. Under a perpetual setting, the investors' optimal stopping rules are characterized by their respective exercise thresholds, which can be quickly determined in a general class of spectrally negative Lévy credit risk models under the completely monotone Lévy density assumption.

For future research, it is most natural to consider the default swap game under a finite horizon and/or different credit risk models. The default swap game studied in this paper can be applied to approximate its finite-maturity version using the maturity randomization (Canadization) approach (see [11, 25]). Another interesting extension is to allow for multiple adjustments by the buyer and/or seller prior to default. This can be modeled as stochastic games with multiple stopping opportunities, leading to more complicated optimal exercise strategies. Finally, the step-up/down feature can also be applied to equity and interest rate derivatives.

Appendix A. Proofs

Proof of Lemma 3.2. Recall that v is given by the first expectation of (2.10), and note that $\sigma_A \wedge \tau_B = \infty$ implies $\theta = \infty$. For every $x \in (A, B)$, we have

$$
\begin{aligned}
&\Delta_h(x; A, B) = v(x; A, B) - h(x) \\
&= \mathbb{E}^x \left[1_{\{\sigma_A \wedge \tau_B < \infty\}} \left(\int_{\sigma_A \wedge \tau_B}^{\theta} e^{-rt} \tilde{p} \, \mathrm{d}t - e^{-r\theta} \tilde{\alpha} 1_{\{\sigma_A \wedge \tau_B < \theta\}} + e^{-r(\sigma_A \wedge \tau_B)} \left(-\gamma_b 1_{\{\tau_B < \sigma_A\}} + \gamma_s 1_{\{\tau_B > \sigma_A\}} \right) \right) \right] \\
&\qquad - \mathbb{E}^x \left[\int_0^{\theta} e^{-rt} \tilde{p} \, \mathrm{d}t \, - e^{-r\theta} \tilde{\alpha} \right] + \gamma_b \\
&= \mathbb{E}^x \left[1_{\{\sigma_A \wedge \tau_B < \infty\}} \left(- \int_0^{\sigma_A \wedge \tau_B} e^{-rt} \tilde{p} \, \mathrm{d}t + e^{-r\theta} \tilde{\alpha} 1_{\{\sigma_A \wedge \tau_B = \theta\}} + e^{-r(\sigma_A \wedge \tau_B)} \left(-\gamma_b 1_{\{\tau_B < \sigma_A\}} + \gamma_s 1_{\{\tau_B > \sigma_A\}} \right) \right) \right. \\
&\qquad \left. - 1_{\{\sigma_A \wedge \tau_B = \infty\}} \left(\int_0^{\theta} e^{-rt} \tilde{p} \, \mathrm{d}t \, - e^{-r\theta} \tilde{\alpha} \right) \right] + \gamma_b \\
&= \mathbb{E}^x \left[1_{\{\sigma_A \wedge \tau_B < \infty\}} e^{-r(\sigma_A \wedge \tau_B)} \left(\tilde{\alpha} 1_{\{\sigma_A \wedge \tau_B = \theta\}} - \gamma_b 1_{\{\tau_B < \sigma_A\}} + \gamma_s 1_{\{\tau_B > \sigma_A\}} \right) - \int_0^{\sigma_A \wedge \tau_B} e^{-rt} \tilde{p} \, \mathrm{d}t \right] + \gamma_b \\
&= \mathbb{E}^x \left[1_{\{\sigma_A \wedge \tau_B < \infty\}} e^{-r(\sigma_A \wedge \tau_B)} \left(\tilde{\alpha} 1_{\{\sigma_A \wedge \tau_B = \theta\}} - \gamma_b 1_{\{\tau_B < \sigma_A\}} + \gamma_s 1_{\{\tau_B > \sigma_A\}} + \frac{\tilde{p}}{r} \right) \right] - \frac{\tilde{p}}{r} + \gamma_b \\
&= \mathbb{E}^x \left[1_{\{\sigma_A \wedge \tau_B < \infty\}} e^{-r(\sigma_A \wedge \tau_B)} \left((\tilde{\alpha} - \gamma_s) 1_{\{\sigma_A \wedge \tau_B = \theta\}} + \left(\frac{\tilde{p}}{r} - \gamma_b \right) 1_{\{\tau_B < \sigma_A\}} + \left(\frac{\tilde{p}}{r} + \gamma_s \right) 1_{\{\tau_B > \sigma_A \text{ or } \sigma_A \wedge \tau_B = \theta\}} \right) \right] \\
&\qquad - \frac{\tilde{p}}{r} + \gamma_b \\
&= \Upsilon(x; A, B) - \frac{\tilde{p}}{r} + \gamma_b.
\end{aligned}
$$

Since $g(x) = h(x) + \gamma_s + \gamma_b$ for every $x > 0$, the second claim is immediate. □

Proof of Lemma 3.3. The expressions for Λ_1 and Λ_2 follow directly from the property of the scale function (see, for example, Theorem 8.1 of [23]).

For Λ_3, let N be the Poisson random measure for $-X$ and $\overline{X}$ and $\underline{X}$ be the running maximum and minimum, respectively, of X. By compensation formula (see e.g. Theorem 4.4 of [23]), we have

$$\begin{aligned}\Lambda_3(x;A,B) &= \mathbb{E}^x\left[\int_0^\infty \int_0^\infty N(\mathrm{d}t\times \mathrm{d}u)e^{-rt}1_{\{\overline{X}_{t-}<B,\, \underline{X}_{t-}>A,\, X_{t-}-u<0\}}\right] \\ &= \mathbb{E}^x\left[\int_0^\infty \mathrm{d}t e^{-rt}\int_0^\infty \Pi(\mathrm{d}u)1_{\{\overline{X}_{t-}<B,\, \underline{X}_{t-}>A,\, X_{t-}-u<0\}}\right] \\ &= \int_0^\infty \Pi(\mathrm{d}u)\int_0^\infty \mathrm{d}t\left[e^{-rt}\mathbb{P}^x\{X_{t-}<u, \sigma_A\wedge\tau_B\geq t\}\right].\end{aligned} \tag{A.1}$$

Recall that, as in Theorem 8.7 of [23], the resolvent measure for the spectrally negative Lévy process killed upon exiting $[0,a]$ is given by

$$\int_0^\infty \mathrm{d}t\left[e^{-rt}\mathbb{P}^x\left\{X_{t-}\in \mathrm{d}y, \sigma_0\wedge\tau_a>t\right\}\right] = \mathrm{d}y\left[\frac{W^{(r)}(x)W^{(r)}(a-y)}{W^{(r)}(a)} - W^{(r)}(x-y)\right], \quad y>0.$$

Hence

$$\begin{aligned}\int_0^\infty \mathrm{d}t\left[e^{-rt}\mathbb{P}^x\left\{X_{t-}\in \mathrm{d}y, \sigma_A\wedge\tau_B>t\right\}\right] &= \int_0^\infty \mathrm{d}t\left[e^{-rt}\mathbb{P}^{x-A}\left\{X_{t-}\in \mathrm{d}(y-A), \sigma_0\wedge\tau_{B-A}>t\right\}\right] \\ &= \mathrm{d}y\left[\frac{W^{(r)}(x-A)W^{(r)}(B-y)}{W^{(r)}(B-A)} - W^{(r)}(x-y)\right],\end{aligned}$$

when $y>A$, and it is zero otherwise. Therefore, for $u>A$, we have

$$\begin{aligned}\int_0^\infty \mathrm{d}t\left[e^{-rt}\mathbb{P}^x\{X_{t-}<u, \sigma_A\wedge\tau_B>t\}\right] &= \int_A^{u\wedge B}\mathrm{d}y\left[\frac{W^{(r)}(x-A)W^{(r)}(B-y)}{W^{(r)}(B-A)} - W^{(r)}(x-y)\right] \\ &= \int_0^{(u-A)\wedge B}\mathrm{d}z\left[\frac{W^{(r)}(x-A)W^{(r)}(B-z-A)}{W^{(r)}(B-A)} - W^{(r)}(x-z-A)\right] \\ &= \frac{W^{(r)}(x-A)}{W^{(r)}(B-A)}\int_0^{u\wedge B-A}\mathrm{d}z W^{(r)}(B-z-A) - \int_0^{u\wedge x-A}\mathrm{d}z W^{(r)}(x-z-A)\end{aligned}$$

since $W^{(r)}$ is zero on $(-\infty,0)$. Therefore, we have

$$\Lambda_3(x,A,B) = \int_A^\infty \Pi(\mathrm{d}u)\left[\frac{W^{(r)}(x-A)}{W^{(r)}(B-A)}\int_0^{u\wedge B-A}\mathrm{d}z W^{(r)}(B-z-A) - \int_0^{u\wedge x-A}\mathrm{d}z W^{(r)}(x-z-A)\right].$$

□

Proof of Lemma 3.4. By Theorem 8.1 of [23], we obtain the limits:

$$\lim_{A\downarrow 0}\Lambda_1(x;A,B) = \mathbb{E}^x\left[e^{-r\tau_B}1_{\{\tau_B<\theta,\, \tau_B<\infty\}}\right] \quad \text{and} \quad \lim_{A\downarrow 0}\Lambda_2(x;A,B) = \mathbb{E}^x\left[e^{-r\tau_B}1_{\{\tau_B=\theta,\, \tau_B<\infty\}}\right].$$

By the construction of Λ_3, as seen in (A.1) above, we deduce that

$$\lim_{A\downarrow 0}\Lambda_3(x;A,B) = \mathbb{E}^x\left[e^{-r\tau_B}1_{\{X_{\tau_B}<0, \tau_B<\infty\}}\right] = \mathbb{E}^x\left[e^{-r\tau_B}1_{\{\tau_B=\theta, \tau_B<\infty\}}\right] - \mathbb{E}^x\left[e^{-r\tau_B}1_{\{X_{\tau_B}=0,\, \tau_B<\infty\}}\right].$$

Applying these to the definition (3.10) yields:

$$\Upsilon(x;0+,B) = \Upsilon(x;0,B) - (\tilde{\alpha}-\gamma_s)\,\mathbb{E}^x\left[e^{-r\tau_B}1_{\{X_{\tau_B}=0,\, \tau_B<\infty\}}\right].$$

By [23] Exercise 7.6, any spectrally negative Lévy process creeps downward, or $\mathbb{P}\{X_\theta = 0 \,|\theta < \infty\} > 0$, if and only if there is a Gaussian component. This completes the proof. $\square$

Proof of Lemma 3.5. (1) The monotonicity is clear because $\partial\kappa(x;A)/\partial A = -W^{(r)}(x-A)\Pi(A,\infty) < 0$ for any $x > A > 0$. (2) By (3.5), we have

$$\int_0^{u\wedge x-A} \mathrm{d}z W^{(r)}(x-z-A) = \int_0^{u\wedge x-A} \mathrm{d}z e^{\Phi(r)(x-z-A)} W_{\Phi(r)}(x-z-A)$$
$$\leq \frac{1}{\phi'(\Phi(r))}\int_0^{u-A} \mathrm{d}z e^{\Phi(r)(x-z-A)} = \frac{e^{\Phi(r)(x-A)}}{\Phi(r)\phi'(\Phi(r))}\left(1-e^{-\Phi(r)(u-A)}\right).$$

Therefore,

$$\kappa(x;A) \leq \frac{e^{\Phi(r)(x-A)}}{\Phi(r)\phi'(\Phi(r))}\rho(A) \leq \frac{e^{\Phi(r)x}}{\Phi(r)\phi'(\Phi(r))}\rho(0). \tag{A.2}$$

Using this with the dominated convergence theorem yields the limit:

$$\kappa(x;0) = \lim_{A\downarrow 0}\frac{1}{r}\int_0^\infty \Pi(\mathrm{d}u+A)\left[Z^{(r)}(x-A) - Z^{(r)}(x-A-u)\right]$$
$$= \frac{1}{r}\int_0^\infty \Pi(\mathrm{d}u)\left[Z^{(r)}(x) - Z^{(r)}(x-u)\right] < \infty.$$

(3) For all $x > A \geq 0$

$$\frac{\kappa(x;A)}{W^{(r)}(x-A)} = \int_A^\infty \Pi(\mathrm{d}u)\int_0^{u\wedge x-A}\mathrm{d}z\frac{W^{(r)}(x-z-A)}{W^{(r)}(x-A)} \leq \int_A^\infty \Pi(\mathrm{d}u)\int_0^{u\wedge x-A} e^{-\Phi(r)z}\mathrm{d}z \leq \frac{\rho(A)}{\Phi(r)}.$$

Therefore, the dominated convergence theorem yields the limit:

$$\lim_{x\uparrow\infty}\frac{\kappa(x;A)}{W^{(r)}(x-A)} = \frac{1}{r}\int_A^\infty \Pi(\mathrm{d}u)\lim_{x\uparrow\infty}\frac{Z^{(r)}(x-A)-Z^{(r)}(x-u)}{W^{(r)}(x-A)} = \frac{\rho(A)}{\Phi(r)}$$

where the last equality holds by (3.7), $Z^{(r)}(x-A)/W^{(r)}(x-A) \xrightarrow{x\uparrow\infty} r/\Phi(r)$ and

$$\lim_{x\uparrow\infty}\frac{Z^{(r)}(x-u)}{W^{(r)}(x-A)} = \lim_{x\uparrow\infty} e^{-\Phi(r)(u-A)}\frac{Z^{(r)}(x-u)}{W^{(r)}(x-u)}\frac{W_{\Phi(r)}(x-u)}{W_{\Phi(r)}(x-A)} = e^{-\Phi(r)(u-A)}\frac{r}{\Phi(r)}.$$

$\square$

Proof of Lemma 3.6. Fix $B > 0$. We have

$$\int_0^\infty \Pi(\mathrm{d}u)\left(1-\frac{W^{(r)}(B-u)}{W^{(r)}(B)}\right) = \Pi(B,\infty) + \frac{1}{W^{(r)}(B)}\int_0^B \Pi(\mathrm{d}u)\left(W^{(r)}(B)-W^{(r)}(B-u)\right) \tag{A.3}$$

where the second term on the right-hand side equals for any $0 < \varepsilon < B$

$$\frac{e^{\Phi(r)B}}{W^{(r)}(B)}\left(\int_0^B \Pi(\mathrm{d}u)W_{\Phi(r)}(B)\left[1-e^{-\Phi(r)u}\right] + \int_0^B \Pi(\mathrm{d}u)e^{-\Phi(r)u}\left[W_{\Phi(r)}(B)-W_{\Phi(r)}(B-u)\right]\right)$$
$$\leq \frac{e^{\Phi(r)B}}{W^{(r)}(B)}\left(W_{\Phi(r)}(B)\rho(0) + W_{\Phi(r)}(B)\Pi(\varepsilon,B) + \alpha(B;\varepsilon)\right), \tag{A.4}$$

with $\alpha(B;\varepsilon) := \int_0^\varepsilon \Pi(\mathrm{d}u)\left[W_{\Phi(r)}(B) - W_{\Phi(r)}(B-u)\right]$.

It is now sufficient to show that $\alpha(B;\varepsilon)$ is finite. As in the proof of Theorem 3 of [28], because Π has a completely monotone density, $W_{\Phi(r)}$ is a Bernstein function admitting the form

$$W_{\Phi(r)}(x) = a + \int_0^\infty \left(1 - e^{-xt}\right) G(\mathrm{d}t), \quad x \geq 0,$$

for some $a \geq 0$ and some finite measure G on $(0,\infty)$. Now using (3.3), we write

$$\begin{aligned} \alpha(B;\varepsilon) &= \int_0^\varepsilon \Pi(\mathrm{d}u) \int_0^\infty G(\mathrm{d}t) \left(e^{-(B-u)t} - e^{-Bt}\right) \\ &= \int_0^\infty G(\mathrm{d}t) e^{-Bt} \int_0^\varepsilon \Pi(\mathrm{d}u) \left(e^{ut} - 1\right) \leq \int_0^\infty G(\mathrm{d}t) e^{-(B-\varepsilon)t} \int_0^\varepsilon \Pi(\mathrm{d}u) \left(1 - e^{-ut}\right) \\ &\leq \int_0^\infty G(\mathrm{d}t) e^{-(B-\varepsilon)t} \int_0^\infty \Pi(\mathrm{d}u) \left(1 - e^{-ut}\right) = \int_0^\infty G(\mathrm{d}t) e^{-(B-\varepsilon)t} \left(\mu t + \frac{\nu^2 t^2}{2} - \phi(t)\right). \end{aligned} \tag{A.5}$$

Since $\mu t + \nu^2 t^2/2 - \phi(t) \sim \nu^2 t^2/2$ as $t \uparrow \infty$ by Exercise 7.6 of [23] and G is a finite measure, this is indeed finite. □

Proof of Lemma 3.7. (1) In view of (3.25), it is immediate by Lemma 3.5-(3) and (3.7). (2) By Lemma 3.5-(2) and because $\rho(A) \xrightarrow{A \downarrow 0} \rho(0)$, the convergence indeed holds. (3) By (A.2), the dominated convergence theorem yields

$$\lim_{B \downarrow A} \Psi(A,B) = \lim_{B \downarrow A} \left[\left(\frac{\tilde{p}}{r} - \gamma_b\right) - \left(\frac{\tilde{p}}{r} + \gamma_s\right) Z^{(r)}(B-A) + (\tilde{\alpha} - \gamma_s)\, \kappa(B;A) \right] = -(\gamma_b + \gamma_s) < 0.$$

□

Proof of Lemma 3.8. (1) Suppose $B < \infty$. Because $W^{(r)}(B-u)/W^{(r)}(B-A)$ is increasing in A on $(0,B)$,

$$\frac{\partial}{\partial A} \widehat{\psi}(A,B) = -\left(\tilde{\alpha} - \gamma_s\right) \int_A^B \Pi(\mathrm{d}u) \frac{\partial}{\partial A} \left[\frac{W^{(r)}(B-u)}{W^{(r)}(B-A)}\right] < 0, \quad 0 < A < B,$$

and hence $\widehat{\psi}$ is decreasing in A on $(0,B)$. The result for $B = \infty$ is immediate because $\rho(A)$ is decreasing.

For the convergence result for $B < \infty$ (when $\rho(0) < \infty$), we have

$$\int_A^\infty \Pi(\mathrm{d}u) \left[1 - \frac{W^{(r)}(B-u)}{W^{(r)}(B-A)}\right] \leq \frac{1}{W^{(r)}(B-A)} \int_0^\infty \Pi(\mathrm{d}u) \left[W^{(r)}(B) - W^{(r)}(B-u)\right],$$

which is bounded by Lemma 3.6. Hence by the dominated convergence theorem,

$$\begin{aligned} \lim_{A \downarrow 0} \int_A^\infty \Pi(\mathrm{d}u) \left[1 - \frac{W^{(r)}(B-u)}{W^{(r)}(B-A)}\right] &= \int_0^\infty \lim_{A \downarrow 0} \Pi(\mathrm{d}u + A) \left[1 - \frac{W^{(r)}(B-u-A)}{W^{(r)}(B-A)}\right] \\ &= \int_0^\infty \Pi(\mathrm{d}u) \left[1 - \frac{W^{(r)}(B-u)}{W^{(r)}(B)}\right]. \end{aligned}$$

The convergence result for $B = \infty$ is clear because $\rho(A) \xrightarrow{A \downarrow 0} \rho(0)$.

(2) Suppose $A > 0$. Look at (3.25) and consider the derivative with respect to B,

$$\frac{\partial}{\partial B} \widehat{\psi}(A,B) = -\left(\tilde{\alpha} - \gamma_s\right) \left[\pi(B) \frac{W^{(r)}(0)}{W^{(r)}(B-A)} + \int_A^B \Pi(\mathrm{d}u) \frac{\partial}{\partial B} \frac{W^{(r)}(B-u)}{W^{(r)}(B-A)}\right]$$

where π is the density of Π. Moreover, for all $A < u < B$,

$$\frac{\partial}{\partial B}\frac{W^{(r)}(B-u)}{W^{(r)}(B-A)} = e^{-\Phi(r)(u-A)}\frac{\partial}{\partial B}\frac{W_{\Phi(r)}(B-u)}{W_{\Phi(r)}(B-A)}$$
$$= e^{-\Phi(r)(u-A)}\frac{W'_{\Phi(r)}(B-u)W_{\Phi(r)}(B-A) - W_{\Phi(r)}(B-u)W'_{\Phi(r)}(B-A)}{(W_{\Phi(r)}(B-A))^2},$$

which is positive by (3.8). Therefore, $\widehat{\psi}(A,B)$ is decreasing in B. This result can be extended to $A = 0$ by (1).

For the convergence result for $A > 0$, the dominated convergence theorem yields

$$\lim_{B\to\infty}\int_A^\infty \Pi(\mathrm{d}u)\left(1 - \frac{W^{(r)}(B-u)}{W^{(r)}(B-A)}\right) = \int_A^\infty \Pi(\mathrm{d}u)\lim_{B\to\infty}\left(1 - \frac{W^{(r)}(B-u)}{W^{(r)}(B-A)}\right) = \rho(A),$$

where the last equality holds by (3.4)-(3.5). When $A = 0$, by (A.3)-(A.4), $W^{(r)}(x) \sim e^{\Phi(r)x}/\phi'(\Phi(r))$ as $x \uparrow \infty$, and because the bound of $\alpha(B,\varepsilon)$ in (A.5) is decreasing in B, we can also apply the dominated convergence theorem and obtain the same result.

(3) The derivative of (3.23) can go into the integral by the dominated convergence theorem because $\frac{1}{r}\int_0^\infty \Pi(\mathrm{d}u)\left(Z^{(r)\prime}(B) - Z^{(r)\prime}(B-u)\right) = \int_0^\infty \Pi(\mathrm{d}u)\left(W^{(r)}(B) - W^{(r)}(B-u)\right) < \infty$ by Lemma 3.6. Therefore, the result is immediate. □

Proof of Lemma 3.9. (1) We rewrite (3.31) as $\Gamma(A,B) = \widehat{\Psi}(A,B) - \widehat{\psi}(A,B)\frac{W^{(r)}(B-A)}{W^{(r)\prime}(B-A)}$. We have shown Lemmas 3.7-(1) and 3.8-(2), and $\Phi(r)\widehat{\Psi}(A,\infty) = \widehat{\psi}(A,\infty)$ in view of (3.24)-(3.25). This, together with (3.6), yields the desired limit.

(2) Recall Lemma 3.7-(3). In the case of unbounded variation, since $W^{(r)\prime}(0+) > 0$, $W^{(r)}(0) = 0$ and $|\widehat{\psi}(A,A+)| < \infty$, it follows that $\Gamma(A,A+) = -\infty$. □

Proof of Theorem 3.1. (1) In view of (a)-(c) in subsection 3.2, we shall show that (i) $\Psi(\underline{A},B)$ monotonically increases while (ii) $\Psi(\overline{A},B)$ monotonically decreases in B.

(i) By the assumption $\underline{A} > 0$, we have $\widehat{\psi}(\underline{A},\infty) = 0$. This coupled with the fact that $\widehat{\psi}(\underline{A},B)$ is decreasing in B by Lemma 3.8-(2) shows that $\widehat{\psi}(\underline{A},B) > 0$ or $\psi(\underline{A},B) > 0$ for every $B > \underline{A}$ and hence $\Psi(\underline{A},B)$ is monotonically increasing in B on $(\underline{A},\infty)$ (recall $\psi(\underline{A},B) = \partial\Psi(\underline{A},B)/\partial B$). Furthermore, $\underline{b}(\underline{A}) < \infty$ implies that $\widehat{\Psi}(\underline{A},\infty) > 0$ (note $\widehat{\Psi}(\underline{A},B) > 0 \iff \Psi(\underline{A},B) > 0$). This together with $W^{(r)}(B-\underline{A}) \xrightarrow{B\uparrow\infty} \infty$ implies that $\Psi(\underline{A},B)$ is monotonically increasing in B to $+\infty$.

(ii) Because $\overline{A} \geq \underline{A}$, we obtain $\overline{A} > 0$ and hence $\widehat{\psi}(\overline{A},\overline{A}+) = 0$. This together with the fact that $\widehat{\psi}(\overline{A},B)$ is decreasing in B by Lemma 3.8-(2) shows that $\widehat{\psi}(\overline{A},B) < 0$, or $\psi(\overline{A},B) < 0$, for every $B > \overline{A}$. Consequently, $\Psi(\overline{A},B)$ is monotonically decreasing in B on $(\overline{A},\infty)$. Furthermore, because $\Psi(\overline{A},\overline{A}+) < 0$ by Lemma 3.7-(3), $\Psi(\overline{A},B)$ never up-crosses the level zero.

By (i) and (ii) and the continuity of Ψ and ψ with respect to both A and B, there must exist $A^* \in (\underline{A},\overline{A})$ and $B^* \in (A^*,\infty)$ such that $B^* = \underline{b}(A^*) = \overline{b}(A^*)$ (with $\Psi(A^*,B^*) = \psi(A^*,B^*) = 0$).

(2) Using the same argument as in (1)-(i) above, $\Psi(\underline{A},B)$ is increasing in B on $(\underline{A},\infty)$. Moreover, the assumption $\underline{b}(\underline{A}) = \infty$ means that $-\infty < \Psi(\underline{A},\underline{A}+) \leq \lim_{B\uparrow\infty}\Psi(\underline{A},B) \leq 0$. This together with $W^{(r)}(B-A) \xrightarrow{B\uparrow\infty} \infty$ shows $\widehat{\Psi}(\underline{A},\infty) = 0$. By (3.6) and Lemma 3.9-(1), $\widehat{\psi}(\underline{A},\infty) = 0$ and this implies that $\widehat{\psi}(\underline{A},B) > 0$ for all $B \in (\underline{A},\infty)$ by virtue of Lemma 3.8-(2), and hence $\overline{b}(\underline{A}) = \infty$.

(3) Recall Lemma 3.8-(3). We have $\psi(0, B) > 0$ if and only if $B \in (0, \bar{b}(0))$, and hence $\Psi(0, \cdot)$ attains a global maximum $\Psi(0, \bar{b}(0))$ and it is strictly larger than zero because $\underline{b}(0) < \bar{b}(0)$. Furthermore, $\Psi(\overline{A}, B)$ is monotonically decreasing in B on $(\overline{A}, \infty)$ and $\Psi(\overline{A}, \overline{A}+) < 0$ as in (1)-(ii). This together with the same argument as in (1) shows the result.

(4) First, $\overline{A} = 0$ implies $\bar{b}(0) = 0$. This also means that $\widehat{\psi}(0, B) \leq 0$ or $\Psi(0, B)$ is decreasing on $(0, \infty)$. This together with Lemma 3.7-(3) shows $\underline{b}(0) = \infty$. Now, for both (i) and (ii) for every $B \in [\bar{b}(0), \underline{b}(0)]$, because $\psi(0, B) \leq 0$, we must have $\Gamma(0, B) \geq \widehat{\Psi}(0, B)$. This shows that $b(0) \leq \underline{b}(0)$. It is clear that this is **case 3** when $b(0) < \infty$ whereas this is **case 4** when $b(0) = \infty$. □

Proof of Lemma 3.10. (1) With $W^{(r)}(B - A) > 0$, it is sufficient to show $\Psi(A, B)$ is decreasing in A on $(\underline{A}, \overline{A})$ for every fixed B. Indeed, the derivative

$$\begin{aligned}\frac{\partial}{\partial A}\Psi(A, B) = \frac{\partial}{\partial A}\left[-\left(\frac{\tilde{p}}{r} + \gamma_s\right) Z^{(r)}(B - A) + (\tilde{\alpha} - \gamma_s)\kappa(B; A)\right] \\ = W^{(r)}(B - A)\left(\tilde{p} + r\gamma_s - (\tilde{\alpha} - \gamma_s)\Pi(A, \infty)\right) \quad \text{(A.6)}\end{aligned}$$

is negative for every $A \in (0, \overline{A})$ by the definition of $\overline{A}$. Part (2) is immediate from Lemma 3.8-(1). □

Proof of Lemma 3.11. (1) Fix $B^* > x > A > A^* > 0$. First, suppose $B^* < \infty$. We compute the derivative:

$$\begin{aligned}\frac{\partial}{\partial A}\Delta_g(x; A, B^*) &= \frac{\partial}{\partial A}\Upsilon(x; A, B^*) = \left[\frac{\partial}{\partial A}\frac{W^{(r)}(x - A)}{W^{(r)}(B^* - A)}\right]\Psi(A, B^*) \\ &+ \frac{W^{(r)}(x - A)}{W^{(r)}(B^* - A)}\left[\frac{\partial}{\partial A}\Psi(A, B^*)\right] + \frac{\partial}{\partial A}\left[\left(\frac{\tilde{p}}{r} + \gamma_s\right) Z^{(r)}(x - A) - (\tilde{\alpha} - \gamma_s)\,\kappa(x; A)\right].\end{aligned}$$

Using (A.6), the last two-terms of the above cancel out and

$$\frac{\partial}{\partial A}\Delta_g(x; A, B^*) = \left[\frac{\partial}{\partial A}\frac{W^{(r)}(x - A)}{W^{(r)}(B^* - A)}\right]\Psi(A, B^*).$$

On the right-hand side, the derivative is given by

$$\begin{aligned}\frac{\partial}{\partial A}\frac{W^{(r)}(x - A)}{W^{(r)}(B^* - A)} &= e^{-\Phi(r)(B^* - x)}\frac{\partial}{\partial A}\frac{W_{\Phi(r)}(x - A)}{W_{\Phi(r)}(B^* - A)} \\ &= e^{-\Phi(r)(B^* - x)}\frac{-W'_{\Phi(r)}(x - A)W_{\Phi(r)}(B^* - A) + W_{\Phi(r)}(x - A)W'_{\Phi(r)}(B^* - A)}{W_{\Phi(r)}(B^* - A)^2}\end{aligned}$$

which is negative according to (3.8) by $B^* > x$. Now suppose $B^* = \infty$. We have

$$\frac{\partial}{\partial A}\Delta_g(x; A, \infty) = \frac{\partial}{\partial A}\left[W^{(r)}(x - A)\widehat{\Psi}(A, \infty)\right] + \frac{\partial}{\partial A}\left[\left(\frac{\tilde{p}}{r} + \gamma_s\right) Z^{(r)}(x - A) - (\tilde{\alpha} - \gamma_s)\,\kappa(x; A)\right].$$

By (3.24), the first term becomes

$$\frac{\partial}{\partial A}\left[W^{(r)}(x - A)\widehat{\Psi}(A, \infty)\right] = -W^{(r)\prime}(x - A)\widehat{\Psi}(A, \infty) - (\widetilde{\alpha} - \gamma_s)W^{(r)}(x - A)\int_A^\infty \Pi(\mathrm{d}u)e^{-\Phi(r)(u - A)},$$

and by using the last equality of (A.6) (with B replaced with x), we obtain

$$-(\widetilde{\alpha}-\gamma_s)W^{(r)}(x-A)\int_A^\infty \Pi(\mathrm{d}u)e^{-\Phi(r)(u-A)}+\frac{\partial}{\partial A}\left[\left(\frac{\tilde{p}}{r}+\gamma_s\right)Z^{(r)}(x-A)-(\tilde{\alpha}-\gamma_s)\,\kappa(x;A)\right]$$
$$=W^{(r)}(x-A)\left(-(\tilde{p}+r\gamma_s)+(\tilde{\alpha}-\gamma_s)\rho(A)\right)=W^{(r)}(x-A)\Phi(r)\widehat{\Psi}(A,\infty).$$

Hence,

$$\frac{\partial}{\partial A}\Delta_g(x;A,\infty)=-\left[W^{(r)\prime}(x-A)-\Phi(r)W^{(r)}(x-A)\right]\widehat{\Psi}(A,\infty)=-e^{\Phi(r)(x-A)}W'_{\Phi(r)}(x-A)\widehat{\Psi}(A,\infty)$$

where $W'_{\Phi(r)}(x-A)>0$ because $W_{\Phi(r)}$ is increasing.

Now in order to show $\Delta_g(x;A,B^*)$ is increasing in A on (A^*,x), it is sufficient to show $\widehat{\Psi}(A,B^*)\le 0$ for every $A^*<A<B^*$. This is true for $A^*<A<\overline{A}$ by $\underline{b}(A^*)=B^*$ and Lemma 3.10-(1). This holds also for $\overline{A}\le A<B^*$. Indeed, $\Psi(A,B)$ is decreasing in B since $\psi(A,B)\le 0$ for any $B>A>\overline{A}$. Furthermore, Lemma 3.7-(3) shows that $\Psi(A,A+)<0$. Hence $\Psi(A,B^*)\le 0$ or $\widehat{\Psi}(A,B^*)\le 0$.

Now we have by (3.28), $0\ge W^{(r)}(0)\widehat{\Psi}(x,B^*)=\Delta_g(x+;x,B^*)>\Delta_g(x;A^*,B^*)$. This proves (3.39) for the case $A^*>0$. Since $\Delta_g(x;0+,B^*)=\lim_{A\downarrow 0}\Delta_g(x;A,B^*)$ by (3.9) and (3.18), this also shows for the case $A^*=0$.

(2) Recall that $\psi(A^*,B)=\partial\Psi(A^*,B)/\partial B$ and hence for any $A^*<x<B<B^*$

$$\begin{aligned}\frac{\partial}{\partial B}\Delta_h(x;A^*,B)&=\frac{\partial}{\partial B}\Upsilon(x;A^*,B)\\&=\frac{W^{(r)}(x-A^*)}{(W^{(r)}(B-A^*))^2}\left[\psi(A^*,B)W^{(r)}(B-A^*)-\Psi(A^*,B)W^{(r)\prime}(B-A^*)\right]\\&=-W^{(r)}(x-A^*)\frac{W^{(r)\prime}(B-A^*)}{W^{(r)}(B-A^*)}\Gamma(A^*,B)\end{aligned}$$

which is positive on (A^*,B^*) by (3.35). Therefore, by (3.27), $0=\Delta_h(x-;A^*,x)\le\Delta_h(x;A^*,B^*)$.

This proves (3.40) for the case $B^*<\infty$. Since $\Delta_h(x;A^*,\infty)=\lim_{B\uparrow\infty}\Delta_h(x;A^*,B)$ by (3.9) and (3.17), this also shows for the case $B^*=\infty$. □

Proof of Lemma 3.12. (1) Suppose $A^*>0$. Because $X_{\sigma_{A^*}\wedge\tau}>A^*$ a.s. on $\{\tau<\sigma_{A^*}\}$, $X_{\sigma_{A^*}\wedge\tau}\le A^*$ a.s. on $\{\tau\ge\sigma_{A^*}\}$ and by (3.40), we have

$$\begin{aligned}g(X_{\sigma_{A^*}\wedge\tau})1_{\{\sigma_{A^*}<\tau\}}+h(X_{\sigma_{A^*}\wedge\tau})1_{\{\tau<\sigma_{A^*}\}}&\le g(X_{\sigma_{A^*}\wedge\tau})1_{\{\sigma_{A^*}<\tau\}}+v_{A^*,B^*}(X_{\sigma_{A^*}\wedge\tau})1_{\{\tau<\sigma_{A^*}\}}\\&=v_{A^*,B^*}(X_{\sigma_{A^*}\wedge\tau})1_{\{\sigma_{A^*}<\tau\}}+v_{A^*,B^*}(X_{\sigma_{A^*}\wedge\tau})1_{\{\tau<\sigma_{A^*}\}}=v_{A^*,B^*}(X_{\sigma_{A^*}\wedge\tau}).\end{aligned}$$

Suppose $A^*=0$. We have by (3.40)

$$-(\tilde{\alpha}-\gamma_s)1_{\{X_\tau=0\}}+h(X_\tau)1_{\{\tau<\theta\}}\le-(\tilde{\alpha}-\gamma_s)1_{\{X_\tau=0\}}+v_{0+,B^*}(X_\tau)1_{\{\tau<\theta\}}=v_{0+,B^*}(X_\tau).$$

The proof for (2) is similar thanks to (3.39). □

Proof of Lemma 3.13. (1) First, Lemma 3.4 of [27] shows that $(\mathcal{L}-r)\zeta(x)=0$. Therefore, using (3.36) and that $J'=J''=0$ on $(0,A^*)$, we have

$$(\mathcal{L}-r)v_{A^*,B^*}(x)=\int_x^\infty\left(J(x-u)-J(x)\right)\Pi(\mathrm{d}u)-rJ(x)=(\tilde{\alpha}-\gamma_s)\,\Pi(x,\infty)-(r\gamma_s+\tilde{p}).\quad\text{(A.7)}$$

Since $A^* > 0$, we must have by construction $\widehat{\Psi}(A^*, B^*) = 0$ and $\Gamma(A^*, B^*) = 0$ and consequently, $\widehat{\psi}(A^*, B^*) = 0$. Furthermore, $\widehat{\psi}(A^*, B)$ is decreasing in B and hence $\widehat{\psi}(A^*, A^*+) = (\tilde{\alpha} - \gamma_s)\,\Pi(A^*, \infty) - (\tilde{p} + \gamma_s r) > 0$. Applying this to (A.7), for $x < A^*$, it follows that $(\mathcal{L} - r)v_{A^*,B^*}(x) > 0$.

(2) When $A^* > 0$, by the strong Markov property,

$$\begin{aligned} &e^{-r(t\wedge\sigma_{A^*}\wedge\tau_{B^*})}v_{A^*,B^*}(X_{t\wedge\sigma_{A^*}\wedge\tau_{B^*}}) \\ &\quad = \mathbb{E}^x\left[e^{-r(\tau_{B^*}\wedge\sigma_{A^*})}\left(h(X_{\tau_{B^*}})1_{\{\tau_{B^*}<\sigma_{A^*}\}} + g(X_{\sigma_{A^*}})1_{\{\tau_{B^*}>\sigma_{A^*}\}}\right)1_{\{\tau_{B^*}\wedge\sigma_{A^*}<\infty\}}\Big|\,\mathcal{F}_{t\wedge\sigma_{A^*}\wedge\tau_{B^*}}\right]. \end{aligned}$$

Taking expectation on both sides, we see that $e^{-r(t\wedge\sigma_{A^*}\wedge\tau_{B^*})}v_{A^*,B^*}(X_{t\wedge\sigma_{A^*}\wedge\tau_{B^*}})$ is a $\mathbb{P}^x$-martingale and hence $(\mathcal{L} - r)v_{A^*,B^*}(x) = 0$ on (A^*, B^*).

When $A^* = 0$ by Lemma 3.4

$$e^{-r(t\wedge\tau_{B^*})}v_{0+,B^*}(X_{t\wedge\tau_{B^*}}) = \mathbb{E}^x\left[e^{-r\tau_{B^*}}\left(h(X_{\tau_{B^*}})1_{\{\tau_{B^*}<\theta\}} - (\tilde{\alpha} - \gamma_s)1_{\{X_{\tau_{B^*}}=0\}}\right)1_{\{\tau_{B^*}<\infty\}}\Big|\,\mathcal{F}_{t\wedge\tau_{B^*}}\right].$$

Taking expectation on both sides, we see that $e^{-r(t\wedge\tau_{B^*})}v_{0+,B^*}(X_{t\wedge\tau_{B^*}})$ is a $\mathbb{P}^x$-martingale and hence $(\mathcal{L} - r)v_{0+,B^*}(x) = 0$ on $(0, B^*)$.

(3) Suppose $\nu > 0$, i.e. there is a Gaussian component. In this case, $W^{(r)}$ is continuous on $\mathbb{R}$ and C^2 on $(0, \infty)$, and we have

$$\begin{aligned} \Delta_h''(B^*-; A^*, B^*) &:= \lim_{x\uparrow B^*}\Delta_h''(x; A^*, B^*) = W^{(r)''}(B^* - A^*)\widehat{\Psi}(A^*, B^*) \\ &\quad + (\tilde{p} + \gamma_s r)W^{(r)'}(B^* - A^*) - (\tilde{\alpha} - \gamma_s)\int_{A^*}^{\infty}\Pi(\mathrm{d}u)\left(W^{(r)'}(B^* - A^*) - W^{(r)'}(B^* - u)\right). \end{aligned}$$

We show $\Delta_h''(B^*-; A^*, B^*) \geq 0$. To this end, we suppose $\Delta_h''(B^*-; A^*, B^*) < 0$ and derive contradiction. The fact that $\Delta_h'(B^*-; A^*, B^*) = 0$ by smooth fit implies that $\Delta_h'(x; A^*, B^*) > 0$ for some $x \in (B^*-\varepsilon, B^*)$. However, since $\Delta_h(B^*-; A^*, B^*) = 0$, this would contradict (3.40). Consequently, $\Delta_h''(B^*-; A^*, B^*) \geq 0$, implying $(\mathcal{L} - r)v_{A^*,B^*}(B^*+) \leq (\mathcal{L} - r)v_{A^*,B^*}(B^*-)$. When $\nu = 0$, $(\mathcal{L} - r)v_{A^*,B^*}(B^*+) = (\mathcal{L} - r)v_{A^*,B^*}(B^*-)$ by continuous and smooth fit.

As a result, for all cases, we conclude that

$$(\mathcal{L} - r)v_{A^*,B^*}(B^*+) \leq (\mathcal{L} - r)v_{A^*,B^*}(B^*-) = 0.$$

Now it is sufficient to show that $(\mathcal{L} - r)v_{A^*,B^*}(x)$ is decreasing on (B^*, ∞). Recall the decomposition (3.36). Because $(\mathcal{L} - r)\zeta(x) = 0$, we shall show $(\mathcal{L} - r)J(x)$ is decreasing on (A^*, B^*).

Now because $J' = J'' = 0$ on $x > B^*$,

$$(\mathcal{L} - r)J(x) = \int_{x-B^*}^{\infty}\Pi(\mathrm{d}u)\left[J(x-u) - \left(\frac{p}{r} - \gamma_b\right)\right] - (p - r\gamma_b), \quad x > B^*.$$

Since $v_{A^*,B^*}(x) \geq h(x)$, we must have that $J(x) \geq \frac{p}{r} - \gamma_b$ on $x < B^*$ (or the integrand of the above is non-negative). This together with the fact that Π has a monotonically decreasing density shows that it is indeed decreasing on (B^*, ∞). ☐

Proof of Theorem 3.2. (1) (i) We show that $v_{A^*,B^*}(x) \geq v(x; \sigma_{A^*}, \tau)$ for every $\tau \in \mathcal{S}$. As is discussed in Remark 2.2, we only need to focus on the set $\mathcal{S}_{A^*}$.

In order to handle the discontinuity of v_{A^*,B^*} at zero, we first construct a sequence of functions $v_n(\cdot)$ such that (a) it is C^2 everywhere except at B^* and A^*, (b) $v_n(x) = v_{A^*,B^*}(x)$ on $x \in (0, \infty)$ and (c)

$v_n(x) \uparrow v_{A^*,B^*}(x)$ pointwise for every fixed $x \in (-\infty, 0)$. Notice that $v_{A^*,B^*}(\cdot)$ is uniformly bounded because $h(\cdot)$ and $g(\cdot)$ are. Hence, we can choose so that v_n is also uniformly bounded for every fixed $n \geq 1$. Because $v'(x; \sigma_{A^*}, \tau_{B^*}) = v_n'(x)$ and $v''_{A^*,B^*}(x) = v_n''(x)$ on $x \in (0,\infty)\backslash\{A^*, B^*\}$ and $v_{A^*,B^*}(x) \geq v_n(x)$ on $(-\infty, 0)$, we have

$$(\mathcal{L} - r)(v_n - v_{A^*,B^*})(x) \leq 0, \quad x \in (0^*, \infty)\backslash\{A^*, B^*\}. \tag{A.8}$$

We have for any $\tau \in S_{A^*}$

$$\mathbb{E}^x \left[\int_0^{\tau \wedge \sigma_{A^*}} e^{-rs} |(\mathcal{L} - r)(v_n - v_{A^*,B^*})(X_{s-})| \mathrm{d}s \right] \leq K \mathbb{E}^x \left[\int_0^{\sigma_{A^*}} e^{-rs} \Pi(X_{s-}, \infty) \mathrm{d}s \right]$$

where $K := \sup_{x \in \mathbb{R}} |v_{A^*,B^*}(x) - v_n(x)| < \infty$ is the maximum difference between v_{A^*,B^*} and v_n. Using N as the Poisson random measure for $-X$ and $\underline{X}$ as the running minimum of X, we have by the compensation formula

$$\begin{aligned}
\mathbb{E}^x \left[\int_0^{\sigma_{A^*}} e^{-rs} \Pi(X_{s-}, \infty) \mathrm{d}s \right] &= \mathbb{E}^x \left[\int_0^\infty \int_0^\infty e^{-rs} 1_{\{\underline{X}_{s-} > A^*, \, u > X_{s-}\}} \Pi(\mathrm{d}u) \mathrm{d}s \right] \\
&= \mathbb{E}^x \left[\int_0^\infty \int_0^\infty e^{-rs} 1_{\{\underline{X}_{s-} > A^*, \, u > X_{s-}\}} N(\mathrm{d}u \times \mathrm{d}s) \right] = \mathbb{E}^x \left[e^{-r\sigma_{A^*}} 1_{\{X_{\sigma_{A^*}} < 0, \, \sigma_{A^*} < \infty\}} \right] < \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}^x \left[\int_0^{\tau \wedge \sigma_{A^*}} e^{-rs} |(\mathcal{L} - r)(v_n - v_{A^*,B^*})(X_{s-})| \mathrm{d}s \right] < \infty, \\
&\int_0^{\tau \wedge \sigma_{A^*}} e^{-rs} |(\mathcal{L} - r)(v_n - v_{A^*,B^*})(X_{s-})| \mathrm{d}s < \infty, \quad \mathbb{P}^x - a.s.,
\end{aligned} \tag{A.9}$$

uniformly for any $n \geq 1$.

By applying Ito's formula to $\{e^{-r(t \wedge \sigma_{A^*})} v_n(X_{t \wedge \sigma_{A^*}}); t \geq 0\}$, we see that

$$\left\{ e^{-r(t \wedge \sigma_{A^*})} v_n(X_{t \wedge \sigma_{A^*}}) - \int_0^{t \wedge \sigma_{A^*}} e^{-rs} \left((\mathcal{L} - r) v_n(X_{s-}) \right) \mathrm{d}s; \quad t \geq 0 \right\} \tag{A.10}$$

is a local martingale. Here, when $\nu > 0$, although v_n is not C^2 at B^* and A^*, the Lebesgue measure of v_n at which $X = B^*$ and $X = A^*$ is zero and hence $v_n''(B^*)$ and $v_n''(A^*)$ can be chosen arbitrarily. For the case of bounded variation, it is not C^1 at A^* but this can be handled in the same way. See also Theorem 2.1 of [33].

Suppose $\{T_k; k \geq 1\}$ is the corresponding localizing sequence, namely,

$$\mathbb{E}^x \left[e^{-r(t \wedge \sigma_{A^*} \wedge T_k)} v_n(X_{t \wedge \sigma_{A^*} \wedge T_k}) \right] = v_n(x) + \mathbb{E}^x \left[\int_0^{t \wedge \sigma_{A^*} \wedge T_k} e^{-rs} \left((\mathcal{L} - r) v_n(X_{s-}) \right) \mathrm{d}s \right], \quad k \geq 1.$$

Now by applying the dominated convergence theorem on the left-hand side and Fatou's lemma on the right-hand side via $(\mathcal{L} - r) v_n(x) \leq 0$ for every $x > 0$ thanks to (A.8) and Lemma 3.13-(2,3), we obtain

$$\mathbb{E}^x \left[e^{-r(t \wedge \sigma_{A^*})} v_n(X_{t \wedge \sigma_{A^*}}) \right] \leq v_n(x) + \mathbb{E}^x \left[\int_0^{t \wedge \sigma_{A^*}} e^{-rs} \left((\mathcal{L} - r) v_n(X_{s-}) \right) \mathrm{d}s \right].$$

Hence (A.10) is a supermartingale.

Now fix $\tau \in \mathcal{S}_{A^*}$. By optional sampling theorem, we have for any $M \geq 0$

$$\begin{aligned}
&\mathbb{E}^x\left[e^{-r(\tau\wedge\sigma_{A^*}\wedge M)}v_n(X_{\tau\wedge\sigma_{A^*}\wedge M})\right]\\
&\leq v_n(x)+\mathbb{E}^x\left[\int_0^{\tau\wedge\sigma_{A^*}\wedge M} e^{-rs}\left((\mathcal{L}-r)v_{A^*,B^*}(X_{s-})+(\mathcal{L}-r)(v_n-v_{A^*,B^*})(X_{s-})\right)\mathrm{d}s\right]\\
&\leq v_n(x)+\mathbb{E}^x\left[\int_0^{\tau\wedge\sigma_{A^*}\wedge M} e^{-rs}\left((\mathcal{L}-r)(v_n-v_{A^*,B^*})(X_{s-})\right)\mathrm{d}s\right],
\end{aligned}$$

where the last inequality holds by Lemma 3.13-(2,3). Applying the dominated convergence theorem on both sides via (A.9), we obtain the inequality:

$$\mathbb{E}^x\left[e^{-r(\tau\wedge\sigma_{A^*})}v_n(X_{\tau\wedge\sigma_{A^*}})1_{\{\tau\wedge\sigma_{A^*}<\infty\}}\right]\leq v_n(x)+\mathbb{E}^x\left[\int_0^{\tau\wedge\sigma_{A^*}} e^{-rs}\left((\mathcal{L}-r)(v_n-v_{A^*,B^*})(X_{s-})\right)\mathrm{d}s\right]. \tag{A.11}$$

We shall take $n\to\infty$ on both sides. For the left-hand side, the dominated convergence theorem implies

$$\lim_{n\to\infty}\mathbb{E}^x\left[e^{-r(\tau\wedge\sigma_{A^*})}v_n(X_{\tau\wedge\sigma_{A^*}})1_{\{\tau\wedge\sigma_{A^*}<\infty\}}\right]=\mathbb{E}^x\left[e^{-r(\tau\wedge\sigma_{A^*})}v_{A^*,B^*}(X_{\tau\wedge\sigma_{A^*}})1_{\{\tau\wedge\sigma_{A^*}<\infty\}}\right].$$

For the right-hand side, we again apply the dominated convergence theorem via (A.9) to get

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}^x\left[\int_0^{\tau\wedge\sigma_{A^*}} e^{-rs}((\mathcal{L}-r)(v_n-v_{A^*,B^*})(X_{s-}))\mathrm{d}s\right]\\
&\qquad=\mathbb{E}^x\left[\lim_{n\to\infty}\int_0^{\tau\wedge\sigma_{A^*}} e^{-rs}((\mathcal{L}-r)(v_n-v_{A^*,B^*})(X_{s-}))\mathrm{d}s\right].
\end{aligned}\tag{A.12}$$

Now fix $\mathbb{P}^x$-a.e. $\omega\in\Omega$. By (A.9) and the dominated convergence theorem,

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^{\tau(\omega)\wedge\sigma_{A^*}(\omega)} e^{-rs}(\mathcal{L}-r)(v_n-v_{A^*,B^*})(X_{s-}(\omega))\mathrm{d}s\\
&\qquad=\int_0^{\tau(\omega)\wedge\sigma_{A^*}(\omega)} e^{-rs}\lim_{n\to\infty}(\mathcal{L}-r)(v_n-v_{A^*,B^*})(X_{s-}(\omega))\mathrm{d}s.
\end{aligned}$$

Finally, since $X_s(\omega)>A^*$ for Lebesgue-a.e. s on $(0,\tau(\omega)\wedge\sigma_{A^*}(\omega))$, and by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty}(\mathcal{L}-r)(v_n-v_{A^*,B^*})(X_{s-}(\omega))=\int_{X_{s-}(\omega)}^{\infty}\Pi(\mathrm{d}u)\lim_{n\to\infty}(v_n(X_{s-}(\omega)-u)-v(X_{s-}(\omega)-u))=0.$$

Hence, the limit (A.12) vanishes, namely,

$$\lim_{n\to\infty}\mathbb{E}^x\left[\int_0^{\tau\wedge\sigma_{A^*}} e^{-rs}((\mathcal{L}-r)(v_n-v_{A^*,B^*})(X_{s-}))\mathrm{d}s\right]=0.$$

Therefore, by taking $n\to\infty$ on both sides of (A.11) (note $v_{A^*,B^*}(x)=v_n(x)$), we have

$$v_{A^*,B^*}(x)\geq\mathbb{E}^x\left[e^{-r(\tau\wedge\sigma_{A^*})}v_{A^*,B^*}(X_{\tau\wedge\sigma_{A^*}})1_{\{\tau\wedge\sigma_{A^*}<\infty\}}\right],\quad \tau\in\mathcal{S}_{A^*}.$$

This inequality and Lemma 3.12-(1) show that $v_{A^*,B^*}(x)\geq v(x;\sigma_{A^*},\tau)$ for any arbitrary $\tau\in\mathcal{S}_{A^*}$.

(ii) Next, we show that $v_{A^*,B^*}(x) \leq v(x;\sigma,\tau_{B^*})$ for every $\sigma \in \mathcal{S}$. Similarly to (i), we only need to focus on the set $\mathcal{S}_{B^*}$. We again use $\{v_n; n \geq 1\}$ defined in (i). Using the same argument as in (i), we obtain

$$\begin{aligned} -\infty < \mathbb{E}^x \left[\int_0^{\sigma\wedge\tau_{B^*}} e^{-rs}(\mathcal{L}-r)(v_n - v_{A^*,B^*})(X_{s-})\mathrm{d}s \right] \leq 0, \\ \int_0^{\sigma\wedge\tau_{B^*}} e^{-rs}|(\mathcal{L}-r)(v_n - v_{A^*,B^*})(X_{s-})|\mathrm{d}s < \infty, \quad \mathbb{P}^x - a.s., \end{aligned} \tag{A.13}$$

uniformly for any $n \geq 1$. Also, for any fixed $\sigma \in \mathcal{S}_{B^*}$, we have by Lemma 3.13-(1,2) that

$$\begin{aligned} &\mathbb{E}^x \left[\int_0^{\sigma\wedge\tau_{B^*}} e^{-rs}((\mathcal{L}-r)v_n(X_{s-}))\mathrm{d}s \right] \\ &= \mathbb{E}^x \left[\int_0^{\sigma\wedge\tau_{B^*}} e^{-rs}\left((\mathcal{L}-r)v_{A^*,B^*}(X_{s-}) + (\mathcal{L}-r)(v_n - v_{A^*,B^*})(X_{s-})\right)\mathrm{d}s \right] \\ &\geq \mathbb{E}^x \left[\int_0^{\sigma\wedge\tau_{B^*}} e^{-rs}(\mathcal{L}-r)(v_n - v_{A^*,B^*})(X_{s-})\mathrm{d}s \right] > -\infty. \end{aligned} \tag{A.14}$$

By applying Ito's formula to $\left\{e^{-r(t\wedge\tau_{B^*})}v_n(X_{t\wedge\tau_{B^*}}); t \geq 0\right\}$, we see that

$$\left\{ e^{-r(t\wedge\tau_{B^*})}v_n(X_{t\wedge\tau_{B^*}}) - \int_0^{t\wedge\tau_{B^*}} e^{-rs}\left((\mathcal{L}-r)v_n(X_{s-})\right)\mathrm{d}s; \quad t \geq 0 \right\} \tag{A.15}$$

is a local martingale. Suppose $\{T_k; k \geq 1\}$ is the corresponding localizing sequence, we have

$$\begin{aligned} &\mathbb{E}^x \left[e^{-r(t\wedge\tau_{B^*}\wedge T_k)}v_n(X_{t\wedge\tau_{B^*}\wedge T_k}) \right] = v_n(x) + \mathbb{E}^x \left[\int_0^{t\wedge\tau_{B^*}\wedge T_k} e^{-rs}\left((\mathcal{L}-r)v_n(X_{s-})\right)\mathrm{d}s \right] \\ &= v_n(x) + \mathbb{E}^x \left[\int_0^{t\wedge\tau_{B^*}\wedge T_k} e^{-rs}\left((\mathcal{L}-r)v_{A^*,B^*}(X_{s-}) + (\mathcal{L}-r)(v_n - v_{A^*,B^*})(X_{s-})\right)\mathrm{d}s \right] \\ &= v_n(x) + \mathbb{E}^x \left[\int_0^{t\wedge\tau_{B^*}\wedge T_k} e^{-rs}\left((\mathcal{L}-r)v_{A^*,B^*}(X_{s-})\right)\mathrm{d}s \right] \\ &\quad + \mathbb{E}^x \left[\int_0^{t\wedge\tau_{B^*}\wedge T_k} e^{-rs}\left((\mathcal{L}-r)(v_n - v_{A^*,B^*})(X_{s-})\right)\mathrm{d}s \right] \end{aligned}$$

where we can split the expectation by (A.14). Now by applying the dominated convergence theorem on the left-hand side and the monotone convergence theorem and the dominated convergence theorem respectively on the two expectations on the right-hand side (using respectively Lemma 3.13-(1,2) and (A.13)), we obtain

$$\mathbb{E}^x \left[e^{-r(t\wedge\tau_{B^*})}v_n(X_{t\wedge\tau_{B^*}}) \right] = v_n(x) + \mathbb{E}^x \left[\int_0^{t\wedge\tau_{B^*}} e^{-rs}\left((\mathcal{L}-r)v_n(X_{s-})\right)\mathrm{d}s \right].$$

Hence (A.15) is a martingale.

Now fix $\sigma \in \mathcal{S}_{B^*}$. By the optional sampling theorem, we have for any $M \geq 0$ using Lemma 3.13-(1,2)

$$\begin{aligned} &\mathbb{E}^x \left[e^{-r(\sigma\wedge\tau_{B^*}\wedge M)}v_n(X_{\sigma\wedge\tau_{B^*}\wedge M}) \right] = v_n(x) + \mathbb{E}^x \left[\int_0^{\sigma\wedge\tau_{B^*}\wedge M} e^{-rs}\left((\mathcal{L}-r)v_n(X_{s-})\right)\mathrm{d}s \right] \\ &= v_n(x) + \mathbb{E}^x \left[\int_0^{\sigma\wedge\tau_{B^*}\wedge M} e^{-rs}\left((\mathcal{L}-r)v_{A^*,B^*}(X_{s-}) + (\mathcal{L}-r)(v_n - v_{A^*,B^*})(X_{s-})\right)\mathrm{d}s \right] \\ &\geq v_n(x) + \mathbb{E}^x \left[\int_0^{\sigma\wedge\tau_{B^*}\wedge M} e^{-rs}\left((\mathcal{L}-r)(v_n - v_{A^*,B^*})(X_{s-})\right)\mathrm{d}s \right]. \end{aligned}$$

Applying the dominated convergence theorem on both sides by (A.13), we have

$$\mathbb{E}^x\left[e^{-r(\sigma\wedge\tau_{B^*})}v_n(X_{\tau\wedge\tau_{B^*}})1_{\{\sigma\wedge\tau_{B^*}<\infty\}}\right]\geq v_n(x)+\mathbb{E}^x\left[\int_0^{\sigma\wedge\tau_{B^*}}e^{-rs}\left((\mathcal{L}-r)(v_n-v_{A^*,B^*})(X_{s-})\right)\mathrm{d}s\right].$$

We can take $n\to\infty$ on both sides along the same line as in (i) and we obtain

$$v_{A^*,B^*}(x)\leq\mathbb{E}^x\left[e^{-r(\sigma\wedge\tau_{B^*})}v_{A^*,B^*}(X_{\sigma\wedge\tau_{B^*}})1_{\{\sigma\wedge\tau_{B^*}<\infty\}}\right],\quad \tau\in\mathcal{S}_{B^*}.$$

This together with Lemma 3.12-(2) shows that $v_{A^*,B^*}(x)\leq v(x;\sigma,\tau_{B^*})$ for any arbitrary $\sigma\in\mathcal{S}_{B^*}$. This completes the proof for (1) (when $A^*>0$).

(2) Suppose $A^*=0$. When $\nu=0$, then the same results as (1)-(i) hold by replacing A^* with 0 and τ_{A^*} with θ. Now suppose $\nu>0$. Using the same argument as in (1) with τ_{A^*} replaced with θ, the supermartingale property of $\left\{e^{-r(t\wedge\theta)}v_{0+,B^*}(X_{t\wedge\theta});t\geq 0\right\}$ holds. This together with Lemma 3.12-(1) shows

$$\begin{aligned}v_{0+,B^*}(x)&\geq\mathbb{E}^x\left[e^{-r\tau}v_{0+,B^*}(X_\tau)1_{\{\tau<\infty\}}\right]\\&\geq\mathbb{E}^x\left[e^{-r\tau}(h(X_\tau)1_{\{\tau<\theta\}}-(\tilde{\alpha}-\gamma_s)1_{\{X_\tau=0\}})1_{\{\tau<\infty\}}\right]=v(x;\sigma_{0+},\tau),\quad\tau\in\mathcal{S}.\end{aligned}$$

As in the proof of Lemma 3.13-(2), $\left\{e^{-r(t\wedge\tau_{B^*})}v_{0+,B^*}(X_{t\wedge\tau_{B^*}});t\geq 0\right\}$ is a martingale. This together with Lemma 3.12-(2) shows that $v_{0+,B^*}(x)\leq v(x;\sigma,\tau_{B^*})$ for all $\sigma\in\mathcal{S}_{B^*}$. Hence (3.37) is established. Finally, $v_{0+,B^*}(x)=\lim_{\varepsilon\downarrow 0}v(x;\sigma_\varepsilon,\tau_{B^*})$ holds because $\Upsilon(x;\varepsilon,B^*)\xrightarrow{\varepsilon\downarrow 0}\Upsilon(x;0+,B^*)$; see in particular Remark 3.2-(2). □

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