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ON SCALE FUNCTIONS OF SPECTRALLY NEGATIVE LÉVY PROCESSES WITH PHASE-TYPE JUMPS

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ABSTRACT. We study the scale function for the class of spectrally negative Lévy processes with phase-type jumps. We consider both the compound Poisson case and the unbounded variation case with diffusion components, and obtain the corresponding scale functions explicitly. Motivated by the fact that the class of phase-type distributions is dense in the class of all positive-valued distributions, we propose a new approach to approximating the scale function for a general spectrally negative Lévy process. We illustrate, in numerical examples, its effectiveness by obtaining the scale functions for Lévy processes with long-tail distributed jumps.

Key words: Phase-type models; Spectrally negative Lévy processes; Scale functions; Wiener-Hopf factorization; Hyperexponential models

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1. INTRODUCTION

Defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X = \{X_t; t \geq 0\}$ be a *spectrally negative* Lévy process of the form

$$X_t - X_0 = \mu t + \sigma B_t - \sum_{n=1}^{N_t} Z_n, \quad 0 \leq t < \infty, \tag{1.1}$$

for some $\mu \in \mathbb{R}$ and $\sigma \geq 0$. Here $B = \{B_t; t \geq 0\}$ is a standard Brownian motion, $N = \{N_t; t \geq 0\}$ is a Poisson process with arrival rate λ, and $Z = \{Z_n; n = 1, 2, \dots\}$ is an i.i.d. sequence of non-negative random variables with density function $f(\cdot)$. These processes are assumed independent. Let $\mathbb{P}^x$ be the (conditional) probability measure under which $X_0 = x$ and we also let $\mathbb{P} \equiv \mathbb{P}^0$. Its *Laplace exponent* is then

$$\psi(s) := \log \mathbb{E}\left[e^{sX_1}\right] = \mu s + \frac{1}{2}\sigma^2 s^2 + \lambda \int_0^\infty (e^{-sz} - 1) f(z) \mathrm{d}z, \quad s \in \mathbb{C}. \tag{1.2}$$

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Associated with every spectrally negative Lévy process, there exists a *(q-)scale function*

$$W^{(q)} : [0, \infty) \mapsto \mathbb{R}, \quad q \geq 0$$

that uniquely solves

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(\beta) - q}, \qquad \beta > \zeta_q \tag{1.3}$$

where

$$\zeta_q := \sup\{s \geq 0 : \psi(s) = q\}, \quad q \geq 0. \tag{1.4}$$

As can be seen in the work of, for example, Bertoin [7, 8], Chaumont [14] and Kyprianou [21], many fluctuation identities concerning spectrally negative Lévy processes can be expressed in terms of scale functions. There are naturally numerous applications in applied probability including, for example, optimal stopping, queuing, branching processes, insurance and credit risk. See Hubalek and Kyprianou [17] and references therein for detailed historical facts and applications of scale functions.

In this paper, we obtain the scale function for the class of Lévy processes with *phase-type* jumps, or Lévy processes in the form (1.1) with Z having phase-type distributions. Consider a continuous-time Markov chain with some initial distribution and state space consisting of a single absorbing state and a finite number of transient states. The phase-type distribution is the distribution of the time to absorption. The class of phase-type distributions includes, for example, the exponential, hyperexponential, Erlang and Coxian distributions; see Section 3 of Asmussen [2].

The phase-type distribution is important owing to its *denseness* in the class of all positive-valued distributions; see Asmussen [1]. By taking advantage of this fact, it is possible to approximate any distribution arbitrarily closely by phase-type distributions. There are a number of existing algorithms for fitting phase-type distributions to a large class of distributions; see, for example, Asmussen [1] for an EM algorithm and Bladt et al. [9] for a Markov chain Monte Carlo approach. Feldmann and Whitt [16] showed that *completely monotone* distributions, including *long-tail* distributions such as the *Pareto* and *Weibull* distributions, can be approximated by hyperexponential distributions, and then proposed an algorithm for fitting hyperexponential distributions to completely monotone distributions. We revisit their work in Section 5.

The class of spectrally negative Lévy processes with phase-type jumps is consequently dense in the class of all spectrally negative Lévy processes as addressed in Proposition 1 of Asmussen et al. [3]. Therefore, at least in principle, the scale function of any spectrally negative Lévy process can be approximated arbitrarily closely by fitting scale functions of Lévy processes with phase-type jumps.

This will be an important tool in applied probability, particularly, in mathematical finance. There have been a number of attempts to introduce jumps to modify the classical Black-Scholes model, which fails to incorporate real-life phenomena such as the volatility smile. Numerous jump types have been considered; examples include the Gaussian model (Merton [24]), the variance-gamma model (Madan et al. [23]), the (generalized) hyperbolic model (Barndorff-Nielsen and Shephard [5]), the normal inverse Gaussian model (Barndorff-Nielsen [4]), the exponential jump diffusion model (Kou and Wang [19, 20]) and the hyperexponential model (Cai [10, 11] and Cai and Kou [12]). Finally, the phase-type model was introduced by Asmussen et al. [3]. The scale functions for these processes can be either obtained or approximated by the results discussed in this paper.

Scale functions for spectrally negative Lévy processes in general do not admit closed-form expressions. Typically, in order to obtain scale functions, one needs to rely on numerical methods. Surya [25] discusses the Laplace inversion algorithm of (1.3) using the Esscher transform. There are, however, a few cases where explicit expressions can be obtained (see Hubalek and Kyprianou [17]), and some analytical properties such as smoothness have been obtained by, for example, Chan et al. [13]. Egami and Yamazaki [15] obtained the scale function for the exponential jump diffusion process by combining the results by Kou and Wang [20] and some fluctuation identities for spectrally negative Lévy processes. In this paper, we generalize their results to obtain the scale function for the general class of Lévy processes with phase-type jumps.

The rest of the paper is organized as in the following. Section 2 considers the spectrally negative Lévy process with phase-type jumps, and then reviews the results by Asmussen et al. [3]. We obtain the scale function for the class of these processes in Section 3, and express it via matrix inversion and address a few examples in Section 4. Section 5 illustrates numerically the approximation of the scale function of a general spectrally negative Lévy process, using examples where jumps are Weibull and Pareto distributed.

2. Spectrally negative Lévy processes with phase-type jumps

We describe in this section the class of spectrally negative Lévy processes with phase-type jumps. We summarize the results from Asmussen et al. [3], focusing on the case when the jumps are only downward (spectrally negative). The main purpose of this section is to present an explicit expression of the Laplace transform

$$\mathbb{E}^x \left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}} \right], \quad q > 0 \tag{2.1}$$

of the *first passage* (or *down-crossing*) time,

$$\tau_a := \inf \{ t \geq 0 : X_t \leq a \}, \quad 0 \leq a < x, \tag{2.2}$$

in terms of the *negative* roots of the *Cramér-Lundberg* equation,

$$\psi(s) = q, \quad q > 0. \tag{2.3}$$

The representation is obtained in Proposition 2.1, and the main idea is to combine it with another representation we address in the next section in order to obtain the scale functions.

2.1. **Phase-type distributions.** Consider a continuous-time Markov chain $Y = \{Y_t; t \geq 0\}$ with finite state space $\{1, \ldots, m\} \cup \{\Delta\}$ where $1, \ldots, m$ are transient and Δ is absorbing. Its initial distribution is given by a simplex $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_m]$ such that $\alpha_i = \mathbb{P}\{Y_0 = i\}$ for every $i = 1, \ldots, m$. The intensity matrix $\boldsymbol{Q}$ is partitioned into the m transient states and the absorbing state Δ, and is given by

$$\boldsymbol{Q} := \begin{bmatrix} \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{0} & 0 \end{bmatrix}.$$

Here $\boldsymbol{T}$ is an $m \times m$-matrix called the phase-type generator, and $\boldsymbol{t} = -\boldsymbol{T}\boldsymbol{1}$ where $\boldsymbol{1} = (1, \ldots, 1)'$ (because each row sums up to zero).

A distribution is called *phase-type* with representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ if it is the distribution of the absorption time to Δ in the Markov chain described above. It is known that $\boldsymbol{T}$ is non-singular and thus invertible; see Asmussen [1]. Its cumulative distribution function (cdf) and probability density function (pdf) are given, respectively, by

$$F(z) = 1 - \boldsymbol{\alpha} e^{\boldsymbol{T}z}\boldsymbol{1} \quad \text{and} \quad f(z) = \boldsymbol{\alpha} e^{\boldsymbol{T}z}\boldsymbol{t}, \quad z \geq 0.$$

Furthermore, the n^{th}-moment for every $n \geq 1$ is

$$\int_0^\infty z^n f(z) \mathrm{d}z = (-1)^n n! \boldsymbol{\alpha} \boldsymbol{T}^{-n} \boldsymbol{1},$$

and the Laplace transform is

$$\hat{\boldsymbol{f}}[s] := \int_0^\infty e^{-sz} f(z) \mathrm{d}z = \boldsymbol{\alpha}(s\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t},$$

which is analytic for every $s \in \mathbb{C}$ except for the eigenvalues of $\boldsymbol{T}$; see Proposition 4.1 of Asmussen et al. [3]. Although a representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ for a phase-type distribution may not be unique, there exists at least one *minimal* representation whose definition is given below.

Definition 2.1 (minimality)**.** A representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ for a distribution function F is called *minimal* if there exists no number $k < m$, k-vector $\boldsymbol{b}$ and $k \times k$-matrix $\boldsymbol{G}$ such that $F(x) = 1 - \boldsymbol{b} e^{\boldsymbol{G}x}\boldsymbol{1}$ for every $x \geq 0$.

We address, in Section 4, that under the minimality condition (and another minor condition), scale functions can be obtained easily via matrix inversion.

2.2. **Spectrally negative Lévy processes with phase-type jumps.** We now consider the process X in the form (1.1) with Z being an i.i.d. sequence of phase-type distributed random variables with representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$. Its Laplace exponent (1.2) becomes

$$\psi(s) = \mu s + \frac{1}{2}\sigma^2 s^2 + \lambda \left(\hat{\boldsymbol{f}}[s] - 1\right) = \mu s + \frac{1}{2}\sigma^2 s^2 + \lambda \left(\boldsymbol{\alpha}(s\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t} - 1\right), \tag{2.4}$$

which is analytic for every $s \in \mathbb{C}$ except for the engenvalues of $\boldsymbol{T}$. We disregard the case when X is a *negative subordinator* (i.e. it is non-increasing a.s.), and consider the following two cases:

Case 1: when $\sigma > 0$ (i.e. X has unbounded variation),

Case 2: when $\sigma = 0$ and $\mu > 0$ (i.e. X is a compound Poisson process).

Notice, in Case 2, that we can write $X_t = U_t - \sum_{n=1}^{N_t} Z_n$ where $U_t = x + \mu t$ is a (positive) subordinator. This implies that down-crossing of a threshold can occur only by jumps; see, for example, Chapter III of Bertoin [7]. On the other hand, in Case 1, down-crossing can occur also by *creeping downward* (by the diffusion components). We need to handle each case separately due to this difference.

Fix $q > 0$. Consider the Cramér-Lundberg equation (2.3) and define the set of (the absolute values of) negative roots:

$$\mathcal{I}_q := \{i : \psi(-\xi_{i,q}) = q \text{ and } \mathcal{R}(\xi_{i,q}) > 0\}.$$

Furthermore, consider the equation $q/(q - \psi(s)) = 0$ and define the set of singularities:

$$\mathcal{J}_q := \left\{ j : \frac{q}{q - \psi(-\eta_j)} = 0 \text{ and } \mathcal{R}(\eta_j) > 0 \right\}.$$

The elements in $\mathcal{I}_q$ and $\mathcal{J}_q$ may not be distinct, and, in this case, we take each as many times as its multiplicity. By Lemma 1-(1) of Asmussen et al. [3], we have

$$|\mathcal{I}_q| = \begin{cases} |\mathcal{J}_q| + 1, & \text{for Case 1,} \\ |\mathcal{J}_q|, & \text{for Case 2.} \end{cases}$$

In particular, if the representation is minimal, we have

$$\left\{ \begin{array}{lll} |\mathcal{I}_q| = m + 1 \quad \text{and} \quad |\mathcal{J}_q| = m, & \text{for Case 1} \\ |\mathcal{I}_q| = m \quad \text{and} \quad |\mathcal{J}_q| = m, & \text{for Case 2} \end{array} \right\}, \tag{2.5}$$

and hence we can set $\mathcal{I}_q = \{1, \ldots, m+1\}$ and $\mathcal{J}_q = \{1, \ldots, m\}$ for Case 1, and $\mathcal{I}_q = \mathcal{J}_q = \{1, \ldots, m\}$ for Case 2.

2.3. **First passage time.** Let κ_q be an independent exponential random variable with parameter $q > 0$ and denote the *running maximum* and *minimum*, respectively, by

$$\overline{X}_t = \sup_{0 \le s \le t} X_s \quad \text{and} \quad \underline{X}_t = \inf_{0 \le s \le t} X_s, \quad t \ge 0.$$

For every $q > 0$, the *Wiener-Hopf factorization* states that

$$\frac{q}{q - \psi(s)} = \varphi_q^+(s)\, \varphi_q^-(s)$$

for every $s \in \mathbb{C}$ such that $\mathcal{R}(s) = 0$, with the *Wiener-Hopf factors*

$$\varphi_q^-(s) := \mathbb{E}\left[\exp(s\underline{X}_{\kappa_q})\right] \quad \text{and} \quad \varphi_q^+(s) := \mathbb{E}\left[\exp(s\overline{X}_{\kappa_q})\right]$$

that are analytic for s with $\mathcal{R}(s) > 0$ and $\mathcal{R}(s) < 0$, respectively.

Owing to Lemma 1 of Asmussen et al. [3], we can obtain the Laplace transform of the first passage time (2.1) explicitly. We have, for every s such that $\mathcal{R}(s) > 0$,

$$\varphi_q^-(s) = \frac{\prod_{j \in \mathcal{J}_q}(s + \eta_j)}{\prod_{j \in \mathcal{J}_q} \eta_j} \frac{\prod_{i \in \mathcal{I}_q} \xi_{i,q}}{\prod_{i \in \mathcal{I}_q}(s + \xi_{i,q})},$$

from which we can obtain the distribution of $\underline{X}_{\kappa_q}$ by the Laplace inverse via partial fraction expansion. When all the roots in $\mathcal{I}_q$ are distinct, we can write

$$\mathbb{P}\left\{-\underline{X}_{\kappa_q} \in \mathrm{d}x\right\} = \sum_{i \in \mathcal{I}_q} A_{i,q} \xi_{i,q} e^{-\xi_{i,q} x} \mathrm{d}x, \quad x > 0 \tag{2.6}$$

where $\{A_{i,q}; i \in \mathcal{I}_q\}$ are the partial fraction coefficients of the expansion,

$$\varphi_q^-(s) - \varphi_q^-(\infty) = \sum_{i \in \mathcal{I}_q} A_{i,q} \frac{\xi_{i,q}}{\xi_{i,q} + s};$$

see Lemma 1 of Asmussen et al. [3]. This can be handled also in the case when the roots are not all distinct. As in Remark 4 of Asmussen et al. [3], let n denote the number of different roots in $\mathcal{I}_q$ and m_i denote the multiplicity of a root $\xi_{i,q}$. Then we have

$$\mathbb{P}\left\{-\underline{X}_{\kappa_q} \in \mathrm{d}x\right\} = \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \xi_{i,q} \frac{(\xi_{i,q} x)^{k-1}}{(k-1)!} e^{-\xi_{i,q} x} \mathrm{d}x, \quad x > 0 \tag{2.7}$$

where

$$A_{i,q}^{(k)} := \frac{1}{(m_i - k)!} \frac{\partial^{m_i - k}}{\partial s^{m_i - k}} \frac{\varphi_q^-(s)(s + \xi_{i,q})^{m_i}}{\xi_{i,q}^k}\bigg|_{s = -\xi_{i,q}}.$$

Now we obtain the Laplace transform of the first passage time (2.1). Notice that, for every $0 \le a < x$,

$$\mathbb{E}^x[e^{-q\tau_a} 1_{\{\tau_a < \infty\}}] = \mathbb{E}[e^{-q\tau_{a-x}} 1_{\{\tau_{a-x} < \infty\}}] = \mathbb{P}\{\tau_{a-x} < \kappa_q\} = \mathbb{P}\left\{\underline{X}_{\kappa_q} < a - x\right\} = \mathbb{P}\left\{-\underline{X}_{\kappa_q} > x - a\right\}$$

where the second equality is obtained by conditioning on the value of τ_{a-x} (with the independence of κ_q and X). This together with (2.6) and (2.7) shows the following.

Proposition 2.1. *We have, for every $0 \leq a < x$,*

$$\mathbb{E}^x \left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}} \right] = \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \xi_{i,q} \int_{x-a}^{\infty} \frac{(\xi_{i,q} y)^{k-1}}{(k-1)!} e^{-\xi_{i,q} y} \mathrm{d}y$$

and, in particular, when the roots are all distinct, we have

$$\mathbb{E}^x \left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}} \right] = \sum_{i \in \mathcal{I}_q} A_{i,q} e^{-\xi_{i,q}(x-a)}.$$

Remark 2.1. *Note that, for later use, we have*

$$\frac{\partial}{\partial a} \mathbb{E}^x \left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}} \right] = \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \xi_{i,q} \frac{(\xi_{i,q}(x-a))^{k-1}}{(k-1)!} e^{-\xi_{i,q}(x-a)}, \quad 0 \leq a < x, \tag{2.8}$$

$$\frac{\partial}{\partial x} \mathbb{E}^x \left[e^{-q\tau_0} 1_{\{\tau_0 < \infty\}} \right] \Big|_{x=0+} = - \sum_{i=1}^{n} A_{i,q}^{(1)} \xi_{i,q} \tag{2.9}$$

where, in particular, when the roots are all distinct, we have

$$\frac{\partial}{\partial a} \mathbb{E}^x \left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}} \right] = \sum_{i \in \mathcal{I}_q} A_{i,q} \xi_{i,q} e^{-\xi_{i,q}(x-a)}, \quad 0 \leq a < x, \tag{2.10}$$

$$\frac{\partial}{\partial x} \mathbb{E}^x \left[e^{-q\tau_0} 1_{\{\tau_0 < \infty\}} \right] \Big|_{x=0+} = - \sum_{i \in \mathcal{I}_q} A_{i,q} \xi_{i,q}. \tag{2.11}$$

3. Scale functions for Lévy processes with phase-type jumps

In this section, we obtain another representation of the Laplace transform of the first passage time (2.1) in terms of the scale function and the unique *positive* root ζ_q as in (1.4) of the *Cramér-Lundberg* equation (2.3). It is then combined with the result in the previous section to obtain the scale function. For a comprehensive account of scale functions, see, for example, Bertoin [7, 8], Kyprianou [21] and Kyprianou and Surya [22].

3.1. **The Laplace transform of the first passage time in terms of scale functions.** We begin with basic properties of the scale function. Recall that τ_a is the first down-crossing time of threshold a as defined in (2.2). We also define the *first up-crossing time* by

$$\tau_b^+ := \inf \{ t \geq 0 : X_t \geq b \}, \quad 0 \leq x < b.$$

Then we have, for every $q \geq 0$ and $0 \leq x < b$,

$$\mathbb{E}^x \left[e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0\}} \right] = \frac{W^{(q)}(x)}{W^{(q)}(b)}$$

and

$$\mathbb{E}^x\left[e^{-q\tau_0}1_{\{\tau_b^+>\tau_0\}}\right]=Z^{(q)}(x)-Z^{(q)}(b)\frac{W^{(q)}(x)}{W^{(q)}(b)} \tag{3.1}$$

where

$$Z^{(q)}(x)=1+q\int_0^x W^{(q)}(y)\mathrm{d}y,\quad x\geq 0.$$

Here, as in the last section, we disregard the case when X is a negative subordinator.

As an extension to (3.1), we have the following (see Theorem 8.1 of Kyprianou [21]).

Lemma 3.1. *For every $q>0$ and $0\leq a<x$, we have*

$$\mathbb{E}^x\left[e^{-q\tau_a}1_{\{\tau_a<\infty\}}\right]=Z^{(q)}(x-a)-\frac{q}{\zeta_q}W^{(q)}(x-a).$$

The following lemma, taken from Corollary 8.3 of Kyprianou [21], implies that the scale function is continuous at $q=0$ for every fixed $x\geq 0$. Because of the continuity at $q=0$, we can obtain the scale function for $q=0$ by taking the limit as $q\to 0$; see Subsection 3.4.

Lemma 3.2. *For each $x\geq 0$, the function $q\to W^{(q)}(x)$ may be analytically extended to $q\in\mathbb{C}$.*

As is discussed in Kyprianou [21] and Surya [25], there exists a "version" of the scale function $W_{\zeta_q}=\{W_{\zeta_q}(x);x\geq 0\}$ that satisfies, for every fixed $q\geq 0$,

$$W^{(q)}(x)=e^{\zeta_q x}W_{\zeta_q}(x),\quad x\geq 0 \tag{3.2}$$

and

$$\int_0^\infty e^{-\beta x}W_{\zeta_q}(x)\mathrm{d}x=\frac{1}{\psi(\beta+\zeta_q)-q},\quad \beta>0. \tag{3.3}$$

Suppose $\mathbb{P}_c$ is the probability measure defined by the Esscher transform

$$\left.\frac{\mathrm{d}\mathbb{P}_c}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}=e^{cX_t-\psi(c)t},\quad t\geq 0$$

where $c>0$ is arbitrary and $\{\mathcal{F}_t;t\geq 0\}$ is the filtration generated by X; see page 78 of Kyprianou [21]. Then W_{ζ_q} under $\mathbb{P}_{\zeta_q}$ is analogous to $W^{(0)}$ under $\mathbb{P}$. Furthermore, it is known that

$$W_{\zeta_q}(x)\sim(\psi'(\zeta_q))^{-1}\quad\text{as } x\to\infty,$$

and hence the scale function $W^{(q)}$ increases exponentially in x;

$$W^{(q)}(x)\sim\frac{e^{\zeta_q x}}{\psi'(\zeta_q)}\quad\text{as } x\to\infty. \tag{3.4}$$

Due to the fact that W_{ζ_q} does not explode for large x as opposed to $W^{(q)}$, it is often convenient to deal with W_{ζ_q} and use (3.2) to convert it to $W^{(q)}$, especially when numerical computations are involved; see Surya [25]. In this paper, we obtain both W_{ζ_q} and $W^{(q)}$.

3.2. **Asymptotic behaviors at zero.** We now pursue additional properties of the scale function focusing on spectrally negative Lévy processes with phase-type jumps. It has been shown by Chan et al. [13] that if a Lévy process has a Gaussian component, we have $W^{(q)} \in C^2(0,\infty)$. When it does not have a Gaussian component and if its jump distribution has no atoms, we have $W^{(q)} \in C^1(0,\infty)$. Therefore the scale function considered here is at least in $C^1(0,\infty)$.

We use the asymptotic behaviors of the scale function in the neighborhood of zero, which can be found in, for example, Lemmas 4.3 and 4.4 of Kyprianou and Surya [22].

Lemma 3.3. *For every $q \geq 0$, we have*

$$W^{(q)}(0) = \left\{ \begin{array}{ll} 0, & \textit{for Case 1} \\ \frac{1}{\mu}, & \textit{for Case 2} \end{array} \right\} \quad \textit{and} \quad W^{(q)\prime}(0+) = \left\{ \begin{array}{ll} \frac{2}{\sigma^2}, & \textit{for Case 1} \\ \frac{q+\lambda}{\mu^2}, & \textit{for Case 2} \end{array} \right\}.$$

Lemma 3.1 and (3.2) imply that, for every $q > 0$,

$$\mathbb{E}^x \left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}} \right] = 1 + q \int_0^{x-a} W^{(q)}(y) \mathrm{d}y - \frac{q}{\zeta_q} e^{\zeta_q (x-a)} W_{\zeta_q}(x-a).$$

Because $W^{(q)} \in C^1(0,\infty)$, its derivative with respect to a becomes

$$\begin{aligned} \frac{\partial}{\partial a} \mathbb{E}^x \left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}} \right] &= -q W^{(q)}(x-a) + q e^{\zeta_q (x-a)} W_{\zeta_q}(x-a) + \frac{q}{\zeta_q} e^{\zeta_q (x-a)} W'_{\zeta_q}(x-a) \\ &= \frac{q}{\zeta_q} e^{\zeta_q (x-a)} W'_{\zeta_q}(x-a). \end{aligned}$$

On the other hand, when $a = 0$, the derivative with respect to x and its limit as $x \to 0$ are

$$\frac{\partial}{\partial x} \mathbb{E}^x \left[e^{-q\tau_0} 1_{\{\tau_0 < \infty\}} \right] = -\frac{q}{\zeta_q} \left[-\zeta_q W^{(q)}(x) + W^{(q)\prime}(x) \right] \xrightarrow{x \downarrow 0+} -\frac{q}{\zeta_q} \left[-\zeta_q W^{(q)}(0) + W^{(q)\prime}(0+) \right] = -\frac{q}{\zeta_q} \theta$$

where, by Lemma 3.3,

$$\theta := -\zeta_q W^{(q)}(0) + W^{(q)\prime}(0+) = \left\{ \begin{array}{ll} \frac{2}{\sigma^2}, & \text{for Case 1} \\ -\frac{\zeta_q}{\mu} + \frac{q+\lambda}{\mu^2}, & \text{for Case 2} \end{array} \right\}. \tag{3.5}$$

In summary, we have the following lemma.

Lemma 3.4. *For every* $q > 0$, *we have*

$$\frac{\partial}{\partial a}\mathbb{E}^x\left[e^{-q\tau_a}1_{\{\tau_a<\infty\}}\right] = \frac{q}{\zeta_q}e^{\zeta_q(x-a)}W'_{\zeta_q}(x-a), \quad 0 \le a < x,$$

$$\frac{\partial}{\partial x}\mathbb{E}^x\left[e^{-q\tau_0}1_{\{\tau_0<\infty\}}\right]\Big|_{x=0+} = -\frac{q}{\zeta_q}\theta.$$

3.3. **Scale functions for Lévy processes with phase-type jumps.** We are now ready to obtain the scale function for the Lévy process with phase-type jumps. We assume in this subsection that $q > 0$ and address in the next subsection for the case $q = 0$. We shall first represent the positive root ζ_q (1.4) of the Cramér-Lundberg equation (2.3) in terms of the negative roots $\{\xi_{i,q};\ i \in \mathcal{I}_q\}$. Recall that A's are those obtained by inverting the Laplace transform of the Wiener-Hopf factor as in (2.6) and (2.7). Let us define

$$\varrho_q := \sum_{i=1}^{n} A_{i,q}^{(1)}\xi_{i,q}, \quad q > 0, \tag{3.6}$$

which reduces, when the representation is minimal, to

$$\varrho_q = \sum_{i\in\mathcal{I}_q} A_{i,q}\xi_{i,q}, \quad q > 0. \tag{3.7}$$

Lemma 3.5. *For every* $q > 0$, *we have*

$$\frac{\zeta_q}{q} = \frac{\theta}{\varrho_q}. \tag{3.8}$$

Proof. By (2.9) and (2.11), we have

$$\frac{\partial}{\partial x}\mathbb{E}^x\left[e^{-q\tau_0}1_{\{\tau_0<\infty\}}\right]\Big|_{x=0+} = -\varrho_q.$$

Matching this and the second claim of Lemma 3.4, we have the claim. □

We now obtain the version of the scale function $W_{\zeta_q}(\cdot)$. In the lemma below, $W_{\zeta_q}(0) = W^{(q)}(0)$ is either 0 or $\frac{1}{\mu}$ depending on if it is Case 1 or Case 2; see Lemma 3.3.

Lemma 3.6. *For every* $q > 0$, *we have*

$$W_{\zeta_q}(x) - W_{\zeta_q}(0) = \frac{\theta}{\varrho_q}\sum_{i=1}^{n}\sum_{k=1}^{m_i} A_{i,q}^{(k)}\left(\frac{\xi_{i,q}}{\zeta_q+\xi_{i,q}}\right)^k\left[1 - e^{-(\zeta_q+\xi_{i,q})x}\sum_{j=0}^{k-1}\frac{((\zeta_q+\xi_{i,q})x)^j}{j!}\right], \quad x \ge 0.$$

In particular, when the roots are all distinct, we have

$$W_{\zeta_q}(x) - W_{\zeta_q}(0) = \sum_{i\in\mathcal{I}_q} C_{i,q}\left[1 - e^{-(\zeta_q+\xi_{i,q})x}\right], \quad x \ge 0$$

where

$$C_{i,q} := \frac{\theta}{\varrho_q} \frac{\xi_{i,q} A_{i,q}}{\zeta_q + \xi_{i,q}}, \quad i \in \mathcal{I}_q. \tag{3.9}$$

Proof. Fix $0 \leq a < x$. By (2.8), we have

$$\frac{\partial}{\partial a} \mathbb{E}^x \left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}} \right] = \sum_{i=1}^n \sum_{k=1}^{m_i} A_{i,q}^{(k)} \xi_{i,q} \frac{(\xi_{i,q}(x-a))^{k-1}}{(k-1)!} e^{-\xi_{i,q}(x-a)}.$$

Matching this and the first claim of Lemma 3.4 and using (3.8), we have

$$\begin{aligned} W'_{\zeta_q}(x-a) &= \frac{\theta}{\varrho_q} \sum_{i=1}^n \sum_{k=1}^{m_i} A_{i,q}^{(k)} \xi_{i,q} \frac{(\xi_{i,q}(x-a))^{k-1}}{(k-1)!} e^{-(\zeta_q + \xi_{i,q})(x-a)} \\ &= \frac{\theta}{\varrho_q} \sum_{i=1}^n \sum_{k=1}^{m_i} A_{i,q}^{(k)} (\zeta_q + \xi_{i,q}) \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \frac{((\zeta_q + \xi_{i,q})(x-a))^{k-1}}{(k-1)!} e^{-(\zeta_q + \xi_{i,q})(x-a)}. \end{aligned}$$

Because $0 \leq a < x$ is arbitrary, we have

$$W'_{\zeta_q}(y) = \frac{\theta}{\varrho_q} \sum_{i=1}^n \sum_{k=1}^{m_i} A_{i,q}^{(k)} (\zeta_q + \xi_{i,q}) \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \frac{((\zeta_q + \xi_{i,q})y)^{k-1}}{(k-1)!} e^{-(\zeta_q + \xi_{i,q})y}, \quad y \geq 0.$$

Integrating the above, we have for every $x \geq 0$

$$\begin{aligned} W_{\zeta_q}(x) - W_{\zeta_q}(0) &= \frac{\theta}{\varrho_q} \sum_{i=1}^n \sum_{k=1}^{m_i} A_{i,q}^{(k)} \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \frac{\zeta_q + \xi_{i,q}}{(k-1)!} \int_0^x ((\zeta_q + \xi_{i,q})y)^{k-1} e^{-(\zeta_q + \xi_{i,q})y} \mathrm{d}y \\ &= \frac{\theta}{\varrho_q} \sum_{i=1}^n \sum_{k=1}^{m_i} A_{i,q}^{(k)} \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \frac{1}{(k-1)!} \int_0^{(\zeta_q + \xi_{i,q})x} z^{k-1} e^{-z} \mathrm{d}z. \end{aligned}$$

The first claim is now immediate because the integral part is a lower incomplete gamma function,

$$\int_0^{(\zeta_q + \xi_{i,q})x} z^{k-1} e^{-z} \mathrm{d}z = (k-1)! \left[1 - e^{-(\zeta_q + \xi_{i,q})x} \sum_{j=0}^{k-1} \frac{((\zeta_q + \xi_{i,q})x)^j}{j!} \right], \quad x \geq 0.$$

In particular, when the roots are all distinct (see (2.10)), we have

$$W_{\zeta_q}(x) - W_{\zeta_q}(0) = \frac{\theta}{\varrho_q} \sum_{i \in \mathcal{I}_q} \frac{\xi_{i,q} A_{i,q}}{\zeta_q + \xi_{i,q}} \left[1 - e^{-(\zeta_q + \xi_{i,q})x} \right], \quad x \geq 0,$$

as desired. □

Lemma 3.6 and (3.4) imply that

$$W^{(q)}(x) = \frac{\theta}{\varrho_q} \sum_{i=1}^n \sum_{k=1}^{m_i} A_{i,q}^{(k)} \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \sum_{j=0}^{k-1} \frac{((\zeta_q + \xi_{i,q})x)^j}{j!} \right] + W_{\zeta_q}(0) e^{\zeta_q x}, \quad x \geq 0$$

where, in particular, when the roots are all distinct,

$$W^{(q)}(x) = \sum_{i \in \mathcal{I}_q} C_{i,q} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \right] + W_{\zeta_q}(0) e^{\zeta_q x}, \quad x \geq 0. \tag{3.10}$$

This together with Lemma 3.3 and (3.5) shows the following.

Proposition 3.1. *For every $q > 0$, we have the following.*

(1) *For Case 1, we have*

$$W^{(q)}(x) = \frac{2}{\sigma^2 \varrho_q} \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \sum_{j=0}^{k-1} \frac{((\zeta_q + \xi_{i,q})x)^j}{j!} \right], \quad x \geq 0$$

where, in particular, when the roots are all distinct,

$$W^{(q)}(x) = \frac{2}{\sigma^2 \varrho_q} \sum_{i \in \mathcal{I}_q} \frac{\xi_{i,q} A_{i,q}}{\zeta_q + \xi_{i,q}} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \right], \quad x \geq 0.$$

(2) *For Case 2, we have*

$$W^{(q)}(x) = \frac{1}{\varrho_q} \left(-\frac{\zeta_q}{\mu} + \frac{q+\lambda}{\mu^2} \right) \sum_{i=1}^{n} \sum_{k=1}^{m_i} A_{i,q}^{(k)} \left(\frac{\xi_{i,q}}{\zeta_q + \xi_{i,q}} \right)^k \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \sum_{j=0}^{k-1} \frac{((\zeta_q + \xi_{i,q})x)^j}{j!} \right] + \frac{1}{\mu} e^{\zeta_q x}, \quad x \geq 0$$

where, in particular, when the roots are all distinct,

$$W^{(q)}(x) = \frac{1}{\varrho_q} \left(-\frac{\zeta_q}{\mu} + \frac{q+\lambda}{\mu^2} \right) \sum_{i \in \mathcal{I}_q} \frac{\xi_{i,q} A_{i,q}}{\zeta_q + \xi_{i,q}} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \right] + \frac{1}{\mu} e^{\zeta_q x}, \quad x \geq 0.$$

Remark 3.1. Suppose the roots are all distinct. By Lemma 3.6, we have, for every $\beta > 0$,

$$\begin{aligned}
\int_0^\infty e^{-\beta x} W_{\zeta_q}(x) \mathrm{d}x &= \sum_{i \in \mathcal{I}_q} C_{i,q} \int_0^\infty e^{-\beta x} \left[1 - e^{-(\zeta_q + \xi_{i,q})x} \right] \mathrm{d}x + W_{\zeta_q}(0) \int_0^\infty e^{-\beta x} \mathrm{d}x \\
&= \sum_{i \in \mathcal{I}_q} C_{i,q} \left(\frac{1}{\beta} - \frac{1}{\zeta_q + \xi_{i,q} + \beta} \right) + \frac{W_{\zeta_q}(0)}{\beta}.
\end{aligned}$$

Therefore, by (3.3), we have

$$\sum_{i \in \mathcal{I}_q} C_{i,q} \left(\frac{1}{\beta} - \frac{1}{\zeta_q + \xi_{i,q} + \beta} \right) + \frac{W_{\zeta_q}(0)}{\beta} = \frac{1}{\psi(\beta + \zeta_q) - q}, \qquad \beta > 0. \tag{3.11}$$

Similarly, by (3.10), we have, for every $\beta > \zeta_q$,

$$\begin{aligned}
\int_0^\infty e^{-\beta x} W^{(q)}(x) \mathrm{d}x &= \sum_{i \in \mathcal{I}_q} C_{i,q} \int_0^\infty e^{-\beta x} \left[e^{\zeta_q x} - e^{-\xi_{i,q} x} \right] \mathrm{d}x + W_{\zeta_q}(0) \int_0^\infty e^{-(\beta - \zeta_q)x} \mathrm{d}x \\
&= \sum_{i \in \mathcal{I}_q} C_{i,q} \left(\frac{1}{\beta - \zeta_q} - \frac{1}{\xi_{i,q} + \beta} \right) + \frac{W_{\zeta_q}(0)}{\beta - \zeta_q}.
\end{aligned}$$

Therefore, by (1.3), we have

$$\sum_{i \in \mathcal{I}_q} C_{i,q} \left(\frac{1}{\beta - \zeta_q} - \frac{1}{\xi_{i,q} + \beta} \right) + \frac{W_{\zeta_q}(0)}{\beta - \zeta_q} = \frac{1}{\psi(\beta) - q}, \qquad \beta > \zeta_q.$$

These identities may be used alternatively to obtain the parameters $\{C_{i,q}; i \in \mathcal{I}_q\}$. In fact, in Section 5, we will use these to verify the accuracy of the numerically-obtained scale functions.

3.4. **Extensions to** $q = 0$**.** As observed in Lemma 3.2, the scale function for $q = 0$ can be obtained by taking the limit as $q \to 0$ for every fixed $x \geq 0$. Because the Laplace exponent ψ is analytic everywhere except at the eigenvalues of $\boldsymbol{T}$, we can take

$$\xi_{i,q} \xrightarrow{q \downarrow 0} \xi_{i,0}, \quad i \in \mathcal{I}_q,$$

for some $\xi_{i,0}$ such that $\psi(-\xi_{i,0}) = 0$ and $\mathcal{R}(\xi_{i,q}) \geq 0$. We can therefore define the set $\mathcal{I}_0$ (with the value zero included) in the analogous way with the same cardinality as $\mathcal{I}_q$ for $q > 0$. Similarly, there exists ζ_0 as a limit of ζ_q as $q \to 0$.

Unfortunately, when the overall drift

$$\overline{u} := \mathbb{E} X_1 = \psi'(0+)$$

is negative, the representations of scale functions for $q = 0$ in Lemma 3.6 and Proposition 3.1 cannot be obtained directly by substituting the limits of A's and ϱ_q because these values vanish in the limit. Let us take a look at this closer. It is well-known that

$$\zeta_q \xrightarrow{q \downarrow 0} \zeta_0 \left\{ \begin{array}{ll} > 0, & \overline{u} < 0 \\ = 0, & \overline{u} \geq 0 \end{array} \right\}. \tag{3.12}$$

Now in view of (3.8), we must have $\lim_{q \to 0} \varrho_q = 0$ when $\overline{u} < 0$ because the left-hand side of (3.8) tends to infinity as $q \to 0$. Moreover, due to the definition of ϱ_q in (3.6) and (3.7), the limit values of $(A_{i,q}\xi_{i,q})$ for each $i \in \mathcal{I}_q$ must converge to 0 as well (i.e. at least one of $A_{i,q}$ and $\xi_{i,q}$ vanishes in the limit). We therefore need to rely on l'Hôpital's rule in taking the limits. It should be mentioned, however, that in the case $\overline{u} > 0$, the scale functions can be obtained directly by substituting the limits of the roots $\xi_{i,q}$ and ζ_q.

Suppose $\xi_{1,0}$ is the smallest value of $\xi_{i,0}$ in $\mathcal{I}_0$. Then either ζ_0 or $\xi_{1,0}$ must be 0 because $\psi(0) = 0$. In fact, we have

$$\xi_{1,q} \xrightarrow{q \downarrow 0} \xi_{1,0} \left\{ \begin{array}{ll} = 0, & \overline{u} \leq 0 \\ > 0, & \overline{u} > 0 \end{array} \right\}, \tag{3.13}$$

which complements the convergence result of ζ_q. Therefore, when $q = 0$ and $\overline{u} < 0$, one of the terms in the scale function (which are normally exponential functions of x) is no longer exponential because one of the roots vanishes. Furthermore, when $\overline{u} < 0$ and when $\overline{u} > 0$, respectively, we have

$$\frac{\xi_{1,q}}{q} \xrightarrow{q \downarrow 0} -\frac{1}{\psi'(0+)} = -\frac{1}{\overline{u}} \quad \text{and} \quad \frac{\zeta_q}{q} \xrightarrow{q \downarrow 0} \frac{1}{\psi'(0+)} = \frac{1}{\overline{u}}. \tag{3.14}$$

Therefore, regarding the scale functions for $q = 0$, it is expected that the overall drift $\overline{u}$ comes into play. See, for example, Egami and Yamazaki [15] (Lemma 4.4).

4. Obtaining scale functions via matrix inversion

In this section, we focus on the case when the elements in $\mathcal{I}_q$ are all distinct and the representation is minimal. We express the scale function in a simpler form via matrix inversion without relying on the partial fraction expansion. We then obtain the scale functions for the following special cases: hyperexponential jump diffusion, a standard Brownian motion, exponential jump diffusion and a compound Poisson process with exponential jumps. We see that the scale functions for these processes indeed match the results addressed previously in other papers.

4.1. **Scale functions in terms of matrix inversion.** We assume that the roots in $\mathcal{I}_q$ are all distinct and the representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ is minimal. Recall that, by (2.5), we can set $\mathcal{I}_q = \{1, \ldots, m+1\}$ for Case 1 and $\mathcal{I}_q = \{1, \ldots, m\}$ for Case 2. Consider the first passage time τ_a defined in (2.2), and let $G_0, G_1, \ldots, G_m$ be such that G_0 is the event of down-crossing the threshold a by creeping downward ($X_{\tau_a} = a$), and G_j, for every $j = 1, \ldots, m$, is the event caused by a jump of phase-type j; see Asmussen et al. [3] for the formal definition of these events. As discussed earlier, G_0 cannot occur in Case 2.

Fix $q > 0$ and let $\boldsymbol{\pi} = [\pi_1, \ldots, \pi_{|\mathcal{I}_q|}]'$ be a vector consisting of

$$\pi_j = \left\{ \begin{array}{ll} \mathbb{E}^x \left[e^{-q\tau_a} 1_{G_{j-1}} \right], & \text{for Case 1} \\ \mathbb{E}^x \left[e^{-q\tau_a} 1_{G_j} \right], & \text{for Case 2} \end{array} \right\}, \quad j = 1, \ldots, |\mathcal{I}_q|.$$

Let $\widetilde{\boldsymbol{f}}(s)$ be an $|\mathcal{I}_q|$-dimensional row vector of a function such that, in Case 1, its first component is 1 and the other components are $((s\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t})'$, while, in Case 2, it is $((s\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t})'$. Then we have

$$\widetilde{\boldsymbol{f}}(-\xi_{i,q}) = \left\{ \begin{array}{ll} \left[1 \quad \left((-\xi_{i,q}\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t} \right)' \right], & \text{for Case 1} \\ \left((-\xi_{i,q}\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t} \right)', & \text{for Case 2} \end{array} \right\}, \quad i \in \mathcal{I}_q.$$

Fix $0 \le a < x$. By Proposition 2-(2) of Asmussen et al. [3], we have

$$\widetilde{\boldsymbol{f}}(-\xi_{i,q})\boldsymbol{\pi} = e^{-\xi_{i,q}(x-a)}, \quad i \in \mathcal{I}_q. \tag{4.1}$$

We shall express the system of equations defined in (4.1) by

$$\boldsymbol{H}\boldsymbol{\pi} = \boldsymbol{b} \tag{4.2}$$

where $\boldsymbol{H}$ is an $|\mathcal{I}_q|$-dimensional square matrix and $\boldsymbol{b}$ is an $|\mathcal{I}_q|$-dimensional vector such that

$$\boldsymbol{H} = \begin{bmatrix} \widetilde{\boldsymbol{f}}(-\xi_{1,q}) \\ \vdots \\ \widetilde{\boldsymbol{f}}(-\xi_{|\mathcal{I}_q|,q}) \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} e^{-\xi_{1,q}(x-a)} \\ \vdots \\ e^{-\xi_{|\mathcal{I}_q|,q}(x-a)} \end{bmatrix}.$$

Here $\boldsymbol{H}$ is invertible and the solution to (4.2) is unique because $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ is assumed to be minimal; see Asmussen et al. [3] for proof.

From the definition of $\boldsymbol{\pi}$, we have

$$\mathbb{E}^x \left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}}\right] = \sum_{j=1}^{|\mathcal{I}_q|} \pi_j =: \boldsymbol{\pi}' \boldsymbol{1}. \tag{4.3}$$

We now express (4.3) in terms of a linear combination of $e^{-\xi_{1,q}(x-a)}, \ldots, e^{-\xi_{|\mathcal{I}_q|,q}(x-a)}$, and obtain their coefficients. This can be done by changing it to the form considered for the hyperexponential case in Cai [10]. Indeed, (4.3) is a solution to the trivial linear programming,

$$\text{minimize} \quad \boldsymbol{\pi}'\boldsymbol{1} \quad \text{subject to (4.2)},$$

and its dual problem becomes finding an $|\mathcal{I}_q|$-dimensional vector $\boldsymbol{w} = [w_1, \ldots, w_{|\mathcal{I}_q|}]$ for the following:

$$\text{maximize} \quad \boldsymbol{b}'\boldsymbol{w} \quad \text{subject to } \boldsymbol{H}'\boldsymbol{w} = \boldsymbol{1}.$$

Due to the invertibility of $\boldsymbol{H}'$, there exists a unique $\boldsymbol{w}$ that satisfies

$$\boldsymbol{H}'\boldsymbol{w} = \boldsymbol{1}. \tag{4.4}$$

Therefore, by the lack of duality gap, we have

$$\mathbb{E}^x \left[e^{-q\tau_a} 1_{\{\tau_a < \infty\}}\right] = \sum_{i \in \mathcal{I}_q} w_i e^{-\xi_{i,q}(x-a)}.$$

By replacing A's with the entries of $\boldsymbol{w}$ in Proposition 3.1, we obtain the scale function.

In sum, to find the scale function, one merely needs to identify the entries of $\boldsymbol{H}$ in terms of ξ's (the negative roots of the Cramér-Lundberg equation (2.3)) and the phase-type generator $\boldsymbol{T}$. Then $\boldsymbol{w} = (\boldsymbol{H}')^{-1}\boldsymbol{1}$ are put in the place of A's in Proposition 3.1. Note that ϱ_q is defined in (3.6) or (3.7) and ζ_q can be obtained via Lemma 3.5 or solving the Cramér-Lundberg equation (2.3). We shall now illustrate our results using several examples.

4.2. **The spectrally negative hyperexponential jump diffusion process.** As a special case of the results above, consider the spectrally negative hyperexponential jump diffusion process where the jump term Z is a sequence of i.i.d. random variables with density function

$$f(z) = \sum_{i=1}^{m} \alpha_i \eta_i e^{-\eta_i z}, \quad z \geq 0,$$

for some $0 < \eta_1 < \cdots < \eta_m < \infty$ such that $\sum_{i=1}^m \alpha_i = 1$. Its Laplace exponent (2.4) is then

$$\psi(s) = \mu s + \frac{1}{2}\sigma^2 s^2 + \lambda \left(\sum_{i=1}^{m} \frac{\alpha_i \eta_i}{\eta_i + s} - 1 \right), \tag{4.5}$$

and the overall drift is

$$\overline{u} := \mu - \lambda \sum_{i=1}^{m} \frac{\alpha_i}{\eta_i}.$$

This is a special case of the Lévy process with phase-type jumps with generator

$$\boldsymbol{T}_{hyp} := diag(-\eta_1, \ldots, -\eta_m).$$

The class of hyperexponential jump diffusion processes is an important subset of the processes considered in this paper. There are numerous applications despite their simple structures. For example, as we see in Section 5, approximations of a wide class of spectrally negative Lévy processes can be attained by these processes. See also Cai [10, 11] and Cai and Kou [12] and references therein for more details about the hyperexponential jump diffusion process.

It is easy to see that $-\eta_1, \ldots, -\eta_m$ are the eigenvalues of $\boldsymbol{T}$ and the representation $(m, \boldsymbol{\alpha}, \boldsymbol{T})$ is minimal. Consequently, we can set $\mathcal{I}_q = \{1, \ldots, m+1\}$ for Case 1 and $\mathcal{I}_q = \{1, \ldots, m\}$ for Case 2. We further see that these roots can be ordered in ascending order as in the following lemma.

Lemma 4.1. *Fix $q > 0$. For Case 1, $\mathcal{I}_q$ consists of $\xi_{1,q}, \ldots, \xi_{m+1,q}$ such that*

$$0 < \xi_{1,q} < \eta_1 < \xi_{2,q} < \cdots < \eta_m < \xi_{m+1,q} < \infty,$$

and, for Case 2, it consists of $\xi_{1,q}, \ldots, \xi_{m,q}$ such that

$$0 < \xi_{1,q} < \eta_1 < \xi_{2,q} < \cdots < \xi_{m,q} < \eta_m < \infty.$$

Proof. For Case 1, see the proof of Lemma 2.1 in Cai [10]. It can be shown for Case 2 similarly. For every $i = 2, \ldots, m$, notice from (4.5) that $\psi(-\eta_i+) = \infty$ and $\psi(-\eta_{i-1}-) = -\infty$ and due to its continuity on $(-\eta_i, -\eta_{i-1})$, there must be at least one root to $\psi(s) = q$ on the interval. Moreover, because $\psi(0) = 0$ and $\psi(-\eta_1+) = \infty$, there must be at least one root on $(-\eta_1, 0)$. Now, by (2.5), there must be exactly one root on each interval, and hence the proof is complete. □

Figure 1 illustrates the Laplace exponent functions of hyperexponential jump diffusion processes and their roots of Cramér-Lundberg equation (2.3) when the overall drift $\overline{u}$ is negative and positive for both Case 1 and Case 2. It can be reassured that the numbers of negative roots are indeed $m+1$ and m, respectively, for Case 1 and Case 2. Furthermore, the convergence results of ζ_q and $\xi_{1,q}$ as $q \to 0$ are consistent with (3.12) and (3.13).

In order to obtain its scale function for $q > 0$ or $\overline{u} > 0$, we need to solve the system (4.2). In particular, for Case 1, we have

$$\widetilde{\boldsymbol{f}}(s) \equiv \widetilde{\boldsymbol{f}}_{hyp}(s) := \left[1, \frac{\eta_1}{\eta_1 + s}, \dots \frac{\eta_m}{\eta_m + s}\right]$$

and hence

$$\boldsymbol{H}' \equiv \boldsymbol{H'}_{hyp} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{\eta_1}{\eta_1 - \xi_{1,q}} & \frac{\eta_1}{\eta_1 - \xi_{2,q}} & \cdots & \frac{\eta_1}{\eta_1 - \xi_{m+1,q}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\eta_m}{\eta_m - \xi_{1,q}} & \frac{\eta_m}{\eta_m - \xi_{2,q}} & \cdots & \frac{\eta_m}{\eta_m - \xi_{m+1,q}} \end{bmatrix}.$$

On the other hand, for Case 2, we have

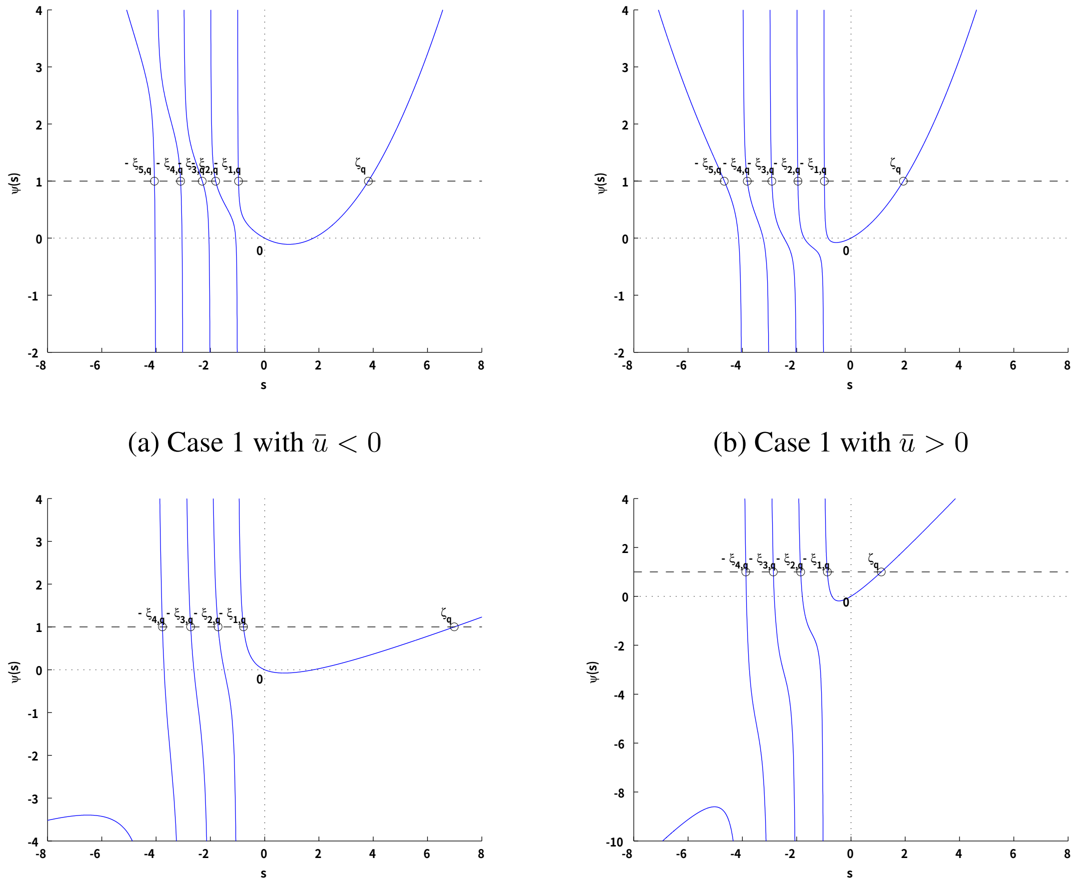
$$\widetilde{\boldsymbol{f}}_{hyp}(s) := \left[\frac{\eta_1}{\eta_1 + s}, \dots \frac{\eta_m}{\eta_m + s}\right]$$

and

$$\boldsymbol{H'}_{hyp} = \begin{bmatrix} \frac{\eta_1}{\eta_1 - \xi_{1,q}} & \frac{\eta_1}{\eta_1 - \xi_{2,q}} & \cdots & \frac{\eta_1}{\eta_1 - \xi_{m,q}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\eta_m}{\eta_m - \xi_{1,q}} & \frac{\eta_m}{\eta_m - \xi_{2,q}} & \cdots & \frac{\eta_m}{\eta_m - \xi_{m,q}} \end{bmatrix}.$$

By inverting $\boldsymbol{H'}_{hyp}$ to obtain $\boldsymbol{w}$ by (4.4) and replacing A's with $\boldsymbol{w}$ in Proposition 3.1, we obtain the scale function; see Section 5 for numerical examples. The case $q = 0$ and $\overline{u} < 0$ can be also handled by differentiating both sides of (4.2) with respect to q as in Cai [10].

4.3. **Other examples.** We consider several special cases of the spectrally negative hyperexponential jump diffusion process. We consider (1) a standard Brownian motion, (2) exponential jump diffusion and (3) a compound Poisson process with exponential jumps, and show that their scale functions can be obtained in the same framework. Here we obtain $W^{(q)}$, which can be converted to W_{ζ_q} by (3.2).



(a) Case 1 with $\bar{u} < 0$

(b) Case 1 with $\bar{u} > 0$

(c) Case 2 with $\bar{u} < 0$

(d) Case 2 with $\bar{u} > 0$

FIGURE 1. Illustrations of the Laplace exponent functions of hyperexponential jump diffusion processes and the roots of the Cramér-Lundberg equation (2.3) when $q = 1$, showing the cases where the overall drift $\overline{u}$ is positive and negative for both Case 1 and Case 2. Here we use parameters $m = 4$, $\eta_1 = 1.0$, $\eta_2 = 2.0$, $\eta_3 = 3.0$ and $\eta_4 = 4.0$ for each case, and (a) $\mu = -0.2$, $\sigma = 0.5$ and $\lambda = 0.1$, (b) $\mu = 0.3$, $\sigma = 0.5$ and $\lambda = 0.1$, (c) $\mu = 0.25$, $\sigma = 0$ and $\lambda = 1$, and (d) $\mu = 1.2$, $\sigma = 0$ and $\lambda = 1$.

Corollary 4.1 (Brownian motion)**.** *For a standard Brownian motion ($\mu = 0$, $\sigma = 1$ and $\lambda = 0$), the scale function is, for every $q > 0$,*

$$W^{(q)}(x) = \sqrt{\frac{2}{q}} \sinh(\sqrt{2q}x), \quad x \geq 0.$$

Proof. Because $m = 0$ and $\sigma = 1 > 0$, we have $\mathcal{I}_q = \{1\}$ and the Laplace exponent becomes

$$\psi(s) = \frac{1}{2}s^2, \quad s \in \mathbb{C}.$$

It is clear that there are one positive root ζ_q and one negative root $-\xi_{1,q}$ of the Cramér-Lundberg equation (2.3) such that

$$\zeta_q = \xi_{1,q} = \sqrt{2q}.$$

We have $\boldsymbol{H} = [1]$ and consequently $\boldsymbol{w} = [1]$. Substituting these in Proposition 3.1, we have the claim. □

We now consider the case jumps are exponentially distributed. The following matches the results obtained in Lemma 4.5 of Egami and Yamazaki [15].

Corollary 4.2 (Exponential jump diffusion)**.** *Suppose $m = 1$, $\sigma > 0$, and jumps are exponentially distributed with parameter $\eta \equiv \eta_1$ (i.e. $\boldsymbol{T} = [-\eta]$). For every $q \geq 0$, we have*

$$W^{(q)}(x) = \frac{2}{\sigma^2} \sum_{i \in \{1,2\}} \frac{l_{i,q}}{\xi_{i,q} + \zeta_q} \left(e^{\zeta_q x} - e^{-\xi_{i,q} x} \right), \quad x \geq 0$$

with

$$l_{1,q} := \frac{\eta - \xi_{1,q}}{\xi_{2,q} - \xi_{1,q}} > 0 \quad \textit{and} \quad l_{2,q} := \frac{\xi_{2,q} - \eta}{\xi_{2,q} - \xi_{1,q}} > 0, \quad q \geq 0.$$

Proof. Because $m = 1$ and $\sigma > 0$, we have $\mathcal{I}_q = \{1, 2\}$. Suppose first that $q > 0$. Note that

$$\boldsymbol{H}' = \begin{bmatrix} 1 & 1 \\ \frac{\eta}{\eta - \xi_{1,q}} & \frac{\eta}{\eta - \xi_{2,q}} \end{bmatrix} \quad \text{and} \quad (\boldsymbol{H}')^{-1} = \frac{(\eta - \xi_{1,q})(\eta - \xi_{2,q})}{\eta(\xi_{2,q} - \xi_{1,q})} \begin{bmatrix} \frac{\eta}{\eta - \xi_{2,q}} & -1 \\ -\frac{\eta}{\eta - \xi_{1,q}} & 1 \end{bmatrix}.$$

Because $\boldsymbol{w} = (\boldsymbol{H}')^{-1} I$, we have

$$\boldsymbol{w} = \frac{(\eta - \xi_{1,q})(\eta - \xi_{2,q})}{\eta(\xi_{2,q} - \xi_{1,q})} \begin{bmatrix} \frac{\eta}{\eta - \xi_{2,q}} & -1 \\ -\frac{\eta}{\eta - \xi_{1,q}} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\eta(\xi_{2,q} - \xi_{1,q})} \begin{bmatrix} \xi_{2,q}(\eta - \xi_{1,q}) \\ \xi_{1,q}(\xi_{2,q} - \eta) \end{bmatrix}.$$

This also implies that

$$\varrho_q = \sum_{i \in \{1,2\}} w_i \xi_{i,q} = \frac{\xi_{1,q}\xi_{2,q}}{\eta},$$

and substituting these in Proposition 3.1, we have the claim. The case for $q = 0$ holds by taking $q \to 0$ as discussed in Subsection 3.4.

□

We now consider, as an example of Case 2, a compound Poisson process with negative exponential jumps. The scale function for this process when $q = 0$ is well-known; see, for example, Hubalek and Kyprianou [17]. Here we obtain its scale function for every $q \geq 0$.

Corollary 4.3 (a compound Poisson process with negative exponential jumps)**.** *Suppose $m = 1$, $\sigma = 0$, and jumps are exponentially distributed with rate $\eta \equiv \eta_1$ (i.e. $\boldsymbol{T} = [-\eta]$). Then, for every $q > 0$, we have*

$$W^{(q)}(x) = \left(-\zeta_q + \frac{q+\lambda}{\mu}\right) \frac{1}{\sqrt{(\eta\mu - q - \lambda)^2 + 4\mu q \eta}} \left[e^{\zeta_q x} - e^{-\xi_{1,q} x}\right] + \frac{1}{\mu} e^{\zeta_q x}, \quad x \geq 0$$

where

$$\begin{aligned} \zeta_q &= \frac{-\eta\mu + q + \lambda + \sqrt{(\eta\mu - q - \lambda)^2 + 4\mu q \eta}}{2\mu}, \\ -\xi_{1,q} &= \frac{-\eta\mu + q + \lambda - \sqrt{(\eta\mu - q - \lambda)^2 + 4\mu q \eta}}{2\mu}. \end{aligned} \tag{4.6}$$

Furthermore, when $q = 0$ and $\overline{u} = \mu - \frac{\lambda}{\eta} \neq 0$, we have

$$W^{(0)}(x) = \frac{1}{\mu} \left[\frac{\lambda}{\eta\mu - \lambda} \left(1 - e^{-\frac{\eta\mu - \lambda}{\mu} x}\right) + 1 \right], \quad x \geq 0,$$

which matches the well-known result as in, for example, Hubalek and Kyprianou [17].

Proof. Because $m = 1$ and $\sigma = 0$, we have $\mathcal{I}_q = \{1\}$ and the Laplace exponent becomes

$$\psi(s) := \mu s + \lambda \left(\frac{\eta}{\eta + s} - 1 \right), \quad s \in \mathbb{C}.$$

For any $q > 0$, by solving the corresponding Cramér-Lundberg equation (2.3), we see that there are two solutions ζ_q and $-\xi_{1,q}$ defined in (4.6). We have $\boldsymbol{H}' = \left[\frac{\eta}{\eta - \xi_{1,q}}\right]$ and therefore $A_{1,q} = \frac{\eta - \xi_{1,q}}{\eta}$ and $\varrho_q = \xi_{1,q} A_{1,q}$. Notice that

$$\zeta_q + \xi_{1,q} = \frac{\sqrt{(\eta\mu - q - \lambda)^2 + 4\mu q \eta}}{\mu}.$$

These together with Proposition 3.1-(2) show the case for $q > 0$.

For the case $q = 0$ and $\overline{u} \neq 0$, by taking the limit as $q \to 0$, we have

$$W^{(0)}(x) = \left\{ \begin{array}{ll} \left(\frac{\lambda}{\mu}\right) \frac{1}{\eta\mu - \lambda} \left[1 - e^{-\xi_{1,0} x}\right] + \frac{1}{\mu}, & \overline{u} > 0 \\ \left(-\zeta_0 + \frac{\lambda}{\mu}\right) \frac{1}{\lambda - \eta\mu} \left[e^{\zeta_0 x} - 1\right] + \frac{1}{\mu} e^{\zeta_0 x}, & \overline{u} < 0 \end{array} \right\}, \quad x \geq 0$$

where

$$\overline{u} = \mu - \frac{\lambda}{\eta}, \quad \zeta_0 = \left\{ \begin{array}{ll} 0, & \overline{u} > 0 \\ \frac{-\eta\mu + \lambda}{\mu}, & \overline{u} < 0 \end{array} \right\} \quad \text{and} \quad \xi_{1,0} = \left\{ \begin{array}{ll} \frac{\eta\mu - \lambda}{\mu}, & \overline{u} > 0 \\ 0, & \overline{u} < 0 \end{array} \right\}.$$

The claim is now immediate after some algebra. □

5. Approximating the scale function of a general spectrally negative Lévy process

We conclude this paper by illustrating a new approach to approximating the scale function for a general spectrally negative Lévy process. As noted earlier, any spectrally negative Lévy process can be approximated arbitrarily closely by fitting phase-type distributions. Here, we use the fitted data computed by Feldmann and Whitt [16] to approximate the scale function when the jumps have Weibull and Pareto distributions. Recall that the Weibull distribution with parameters c and a (denoted Weibull(c, a)) is give by

$$F(t) = 1 - e^{-(t/a)^c}, \quad t \geq 0$$

and the Pareto distribution with positive parameters a and b (denoted Pareto(a,b)) is given by

$$F(t) = 1 - (1 + bt)^{-a}, \quad t \geq 0.$$

These have long-tails, namely

$$e^{\delta t}(1 - F(t)) \to \infty \quad \text{as } t \to \infty$$

for any $\delta > 0$. See Johnson and Kotz [18] for more details about these distributions.

Feldmann and Whitt [16] showed that if a pdf $f(\cdot)$ is *completely monotone*, meaning that all derivatives exist and

$$(-1)^n f^{(n)}(t) \geq 0, \quad t > 0 \text{ and } n \geq 1,$$

then it can be approximated by pdf's of hyperexponential distributions. As shown by Bernstein [6], every completely monotone pdf is a mixture of exponential pdf's, and this implies that, for any cdf with a completely monotone pdf, there exists a sequence of hyperexponential cdf's converging to it. The class of distributions with completely monotone pdf's contains a number of distributions such as the Pareto distribution, the Weibull distribution with $a < 1$, the gamma distribution with parameter less than 1 and the Pareto mixture of exponentials distributions. Feldmann and Whitt [16] took advantage of this fact and proposed a recursive algorithm for fitting hyperexponential distributions to these distributions.

We focus on spectrally negative Lévy processes with Weibull- and Pareto-distributed jumps, and compute their scale functions using the results obtained by Feldmann and Whitt [16]. We consider both the unbounded variation (Case 1) and the compound Poisson (Case 2) cases.

5.1. **Computed scale functions.** Consider the cases where jumps are (i) Weibull-distributed with $c = 0.6$ and $a = 0.665$ and (ii) Pareto-distributed with $a = 1.2$ and $b = 5$. Table 1 shows the parameters of the hyperexponential

i	α_i	η_i
1	0.029931	676.178
2	0.093283	38.7090
3	0.332195	4.27400
4	0.476233	0.76100
5	0.068340	0.24800
6	0.000018	0.09700

(i) Weibull(0.6,0.665)

i	α_i	η_i	i	α_i	η_i
1	8.37E-11	8.3E-09	8	0.000147	0.0020
2	7.18E-10	6.8E-08	9	0.001122	0.0100
3	5.56E-09	3.9E-07	10	0.008462	0.0570
4	4.27E-08	2.2E-06	11	0.059768	0.3060
5	3.27E-07	1.2E-05	12	0.307218	1.5460
6	2.50E-06	6.5E-05	13	0.533823	6.5160
7	1.92E-05	3.5E-04	14	0.089437	23.304

(ii) Pareto(1.2,5)

TABLE 1. Parameters of the hyperexponential distributions fitted to (i) a Weibull distribution with $c = 0.6$ and $a = 0.665$ and (ii) a Pareto distribution with $a = 1.2$ and $b = 5$ (taken from Tables 3 and 9, respectively, of Feldmann and Whitt [16]).

distributions obtained by Feldmann and Whitt [16] fitted to (i) with $m = 6$ and to (ii) with $m = 14$. As can be seen in Fig. 4 and Fig. 9, respectively, for (i) and (ii) in Feldmann and Whitt [16], these fittings are very accurate.

For both (i) and (ii), we consider the Lévy processes with (a) $\mu = 0$, $\sigma = 0.01$ and $\lambda = 0.1$ and (b) $\mu = 0.1$, $\sigma = 0$ and $\lambda = 0.1$ as examples of Case 1 and Case 2, respectively, and compute the corresponding scale functions when $q = 0.2$. The roots $\xi_{\cdot,q}$'s and ζ_q are calculated via the bisection method with error bound $1.0E - 10$. The matrix $\boldsymbol{H}$ is then calculated and A's (or w's) are obtained by matrix inversion as in (4.4). Tables 2 and 3 show A's and the coefficients C's for the scale functions as in (3.9).

5.2. **Verification.** We further verify the accuracy of the obtained scale functions using the identity (3.11). In Table 4, we compare the values of the right- and left-hand sides of (3.11) for various values of β.

For the right-hand side, we used the Laplace exponents corresponding to the fitted hyperexponential jump diffusion processes. One could replace them with those of the Lévy processes with the targeted Weibull- and Pareto-distributed jumps in order to see how close the obtained scale functions are to those for these processes. However, we choose not to do so because the fitting errors are expected to be negligible considering the performance of the fittings as shown in Fig. 4 and Fig. 9 of Feldmann and Whitt [16], and they are expected to be smaller than the computational errors involved in numerically computing the Laplace transforms for these long-tail distributions. In fact, Feldmann and Whitt [16] address that one of their motivations of approximating via hyperexponential distributions comes from the analytical tractability of obtaining their Laplace transforms.

i	$\xi_{i,q}$	$A_{i,q}$	$C_{i,q}$
1	0.0969990705796	0.000010213932094	0.0000049474290
2	0.2387406362121	0.042207380215956	0.0502261535296
3	0.6178972697386	0.181977148543284	0.5577059566440
4	3.7980930145449	0.087513431726977	1.5832236290145
5	37.160241923152	0.051804177184444	6.4758763022306
6	78.497115144071	0.636476073284678	123.22078286003
7	676.26768636481	0.000011575112568	0.0039733229452

(a) $\mu = 0$, $\sigma = 0.01$ and $\lambda = 0.1$

i	$\xi_{i,q}$	$A_{i,q}$	$C_{i,q}$
1	0.0969991162227	0.000009573264242	0.000004473473070
2	0.2398073307540	0.036062429378262	0.039536262104837
3	0.6467831574459	0.143476761697239	0.039536262104837
4	4.0726718323650	0.037351696325079	0.293551992181030
5	38.622287041928	0.001738440970256	0.020866877535261
6	676.14820026674	0.000034414006923	0.000438794874100

(b) $\mu = 0.1$, $\sigma = 0$ and $\lambda = 0.1$

TABLE 2. The computed parameters for Weibull (0.6,0.665).

We see, in Tables 4 and 5, that the computation is very accurate, and the errors caused by the bisection method and matrix inversion are negligible. This verifies the results obtained in this paper and the effectiveness of the approximation method described in this section.

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i	$\xi_{i,q}$	$A_{i,q}$	$C_{i,q}$
1	0.000000008235156	0.007813458670425	0.000000000003246
2	0.000000067941699	0.000849779495869	0.000000000002913
3	0.000000389921386	0.000199758818815	0.000000000003929
4	0.000002199944736	0.000024894827628	0.000000000002763
5	0.000011999924064	0.000006271675736	0.000000000003797
6	0.000064999898911	0.000001541426092	0.000000000005055
7	0.000349996670502	0.000009429635617	0.000000000166497
8	0.001999852975897	0.000072915691854	0.000000007356267
9	0.009994374942093	0.000559383483826	0.000000282006005
10	0.056756368788265	0.004300742193124	0.000012305122823
11	0.296725362741112	0.031481003607834	0.000469432459232
12	1.335002927170950	0.139603262110976	0.009241025235379
13	5.355731274613405	0.150622194007917	0.038035855476823
14	22.605546762702495	0.026354651553805	0.023204214950600
15	78.642108436899349	0.638100712800481	1.248839677422900

(a) $\mu = 0$, $\sigma = 0.01$ and $\lambda = 0.1$

i	$\xi_{i,q}$	$A_{i,q}$	C_i
1	0.000000008235156	0.007813458664207	0.000000000324612
2	0.000000067941699	0.000849779490290	0.000000000291268
3	0.000000389921386	0.000199758811288	0.000000000392946
4	0.000002199944736	0.000024894822336	0.000000000276293
5	0.000011999924064	0.000006271668464	0.000000000379673
6	0.000064999898911	0.000001541416410	0.000000000505441
7	0.000349996670502	0.000009429316546	0.000000016646795
8	0.001999852975897	0.000072901557445	0.000000734891418
9	0.009994403048011	0.000556045657406	0.000027919877439
10	0.056763159011767	0.004157334908929	0.001163016693045
11	0.297941138195587	0.026612250464557	0.035585440075800
12	1.416075105040742	0.081538553045039	0.366437520584395
13	6.134594568717034	0.050061829801001	0.435890183944306
14	23.225218986441462	0.002719941701028	0.029865726745919

(b) $\mu = 0.1$, $\sigma = 0$ and $\lambda = 0.1$

TABLE 3. The computed parameters for Pareto(1.2,5)

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β	LHS	RHS	absolute error
0.05	2636.956051024141	2636.956050828426	1.9571e-007
0.10	1318.038418276205	1318.038418225931	5.0274e-008
0.50	262.9064571661544	262.9064571636729	2.4815e-009
1.00	131.0176256611043	131.0176256603400	7.6429e-010
10.0	12.36501436360020	12.36501436357116	2.9040e-011
100	0.792975691767121	0.792975691766349	7.7205e-013

(a) $\mu = 0$, $\sigma = 0.01$ and $\lambda = 0.1$

β	LHS	RHS	absolute error
0.05	214.3239334009206	214.3239333328580	6.8063e-008
0.10	107.0782263782371	107.0782263613487	1.6888e-008
0.50	21.29602428989161	21.29602428925708	6.3453e-010
1.00	10.58738231829800	10.58738231815195	1.4605e-010
10.0	1.023286182446206	1.023286182446003	2.0317e-013
100	0.100373126437125	0.100373126437134	9.0067e-015

(b) $\mu = 0.1$, $\sigma = 0$ and $\lambda = 0.1$

TABLE 4. The left- and right-hand sides of (3.11) and their differences for Weibull(0.6,0.665).

β	LHS	RHS	absolute error
0.05	2638.718948463406	2638.718947228851	1.2346e-006
0.10	1318.916455287831	1318.916454977145	3.1069e-007
0.50	263.0766634379456	263.0766634248073	1.3138e-008
1.00	131.0994239282779	131.0994239247748	3.5031e-009
10.0	12.36839081902370	12.36839081895586	6.7841e-011
100	0.792429248707535	0.792429248706064	1.4709e-012

(a) $\mu = 0$, $\sigma = 0.01$ and $\lambda = 0.1$

β	LHS	RHS	absolute error
0.05	217.2193200018033	217.2193198697241	1.3208e-007
0.10	108.5313601994563	108.5313601615090	3.7947e-008
0.50	21.59216520759314	21.59216520482761	2.7655e-009
1.00	10.73611494941012	10.73611494847104	9.3908e-010
10.0	1.033125850470901	1.033125850459483	1.1418e-011
100	0.100548132483049	0.100548132483021	2.8005e-014

(b) $\mu = 0.1$, $\sigma = 0$ and $\lambda = 0.1$

TABLE 5. The left- and right-hand sides of (3.11) and their differences for Pareto(1.2,5).